UNIFORM STABILIZATION OF THE EULER-BERNOULLI EQUATION WITH ACTIVE DIRICHLET AND NON-ACTIVE NEUMANN BOUNDARY FEEDBACK CONTROLS

By

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TABLE OF CONTENTS

page

ACKNOWLEDGEMENTS ii

ABSTRACT v

CHAPTERS

1 INTRODUCTION, PRELIMINARIES, STATEMENT OF MAIN RESULTS 1

1.1 Introduction And Literature 1
1.2 Formulation of the Uniform Stabilization Problem and Main Statements 3
Choice of Operators F_1 and F_2 5
Theorem 1.1 7
Theorem 1.2 8
Theorem 1.3 8

2 WELL-POSEDNESS AND STRONG STABILIZATION 10

2.1 Preliminaries, Choice of Stabilizing Feedback 10
2.2 Well-Posedness and Semigroup Generation 14
Lemma 2.5 14
Theorem 2.6 15
Proof of Theorem 2.6 16
Lemma 2.7 18
Proof of Lemma 2.7 19
Corollary 2.8 20
Theorem 2.10 21
Proof of Theorem 2.10 21

3 UNIFORM STABILIZATION 22

3.1 Preliminaries, Change of Variables 22
3.2 Uniform Stabilization 23
Theorem 3.1 23
Theorem 3.2 24
Proof of Theorem 3.2 (A multiplier approach) 24
Lemma 3.3 24
Lemma 3.4 26
Lemma 3.7 29
Lemma 3.8 30
Proof of Lemma 3.8 31

Lemma 3.9 32
Lemma 3.10 32
Lemma 3.11 34
Proof of Theorem 3.1 35

APPENDICES

A BASIC IDENTITIES 37

B TO HANDLE DIFFERENCE OF ENERGY TERM 39

C TO OBTAIN GENERAL IDENTITY 40

REFERENCES 43

BIOGRAPHICAL SKETCH 47

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Jerry Bartolomeo

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Given Ω an open bounded domain in IR^n with sufficiently smooth boundary Γ, we consider the nonhomogeneous Euler-Bernoulli equation in the solution $w(t,x)$:

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q = (0,\infty)\times\Omega \quad (a) \\ w(0,\cdot) = w_0 \ ; \ w_t(0,\cdot) = w_1 & \text{in } \Omega \quad (b) \\ w\Big|_{\Sigma} = g_1 \in L^2(\Sigma) = L^2((0,\infty)\,; L^2(\Gamma)) & \text{on } \Sigma = (0,\infty)\times\Gamma \quad (c) \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Sigma} = g_2 = 0 & \text{on } \Sigma \quad (d) \end{cases} \tag{1}$$

We seek to express the nonzero control function g_1 as as suitable linear feedback applied to the velocity w_t, i.e., $w\Big|_{\Sigma} = Fw_t$, such that $Fw_t \in L^2((0,\infty)\,; L^2(\Gamma))$, and the corresponding closed loop system obtained by using such a feedback in (1) generates a C_0-semigroup which decays uniformly exponentially to zero as $t\to\infty$ in the uniform topology of $Z = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]'$:

$$\| w(t), w_t(t) \|_Z \le Ce^{-\delta t} \| [\, w_0, w_1 \,] \|_Z \ \text{ for all } t \ge 0 \text{ and some } C,\ \delta > 0 \tag{2}$$

Having identified the candidate $Fw_t = -\frac{\partial}{\partial \nu}[\Delta(A^{-3/2}w_t)]$, where A is the operator defined by $Af \doteq \Delta^2 f$; $\mathcal{D}(A) = \left\{ f \in L^2(\Omega) : \Delta^2 f \in L^2(\Omega), \quad f\Big|_\Gamma = \frac{\partial f}{\partial \nu}\Big|_\Gamma = 0 \right\}$, we prove two stabilization results, the second of which (2) is the goal of this thesis and implies the first. However, we include both results to illustrate a contrast in modern day control methods.

Specifically if the domain Ω satisfies a radial vector field assumption

$$(x - x_0) \cdot \nu \geq \gamma > 0 \text{ on } \Gamma \tag{3}$$

where ν is the outward unit normal vector, we prove *strong stabilization*, i.e., solutions go to zero in the strong topology of $\mathcal{Z}$: $\lim_{t \to \infty} \| [w(t), w_t(t)] \|_{\mathcal{Z}} = 0$, by the use of a Hilbert space decomposition for contractive semigroups. Finally, if Ω satisfies (3), we obtain the desired *uniform stabilization* (2) via a change of variables followed by the use of multipliers.

CHAPTER 1
INTRODUCTION, PRELIMINARIES, STATEMENT OF MAIN RESULTS

1.1 Introduction And Literature

Let Ω be an open bounded domain in $I\!R^n$, n typically ≥ 2, with sufficiently smooth boundary Γ. In Ω we consider the Euler-Bernoulli mixed problem in $w(t,x)$ on an arbitrary time interval $(0,T]$:

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } (0,T]\times\Omega \quad (a) \\ w(0,x) = w_0(x)\ ;\ w_t(0,x) = w_1(x) & \text{in } \Omega \quad (b) \\ w(t,\sigma) = g_1(t,\sigma) & \text{on } (0,T]\times\Gamma \quad (c) \\ \dfrac{\partial w}{\partial \nu}(t,\sigma) = g_2(t,\sigma) & \text{on } (0,T]\times\Gamma \quad (d) \end{cases} \tag{1.1}$$

with non-homogeneous forcing terms (control functions g_1 and g_2 in the Dirichlet and Neumann boundary conditions). In (1.1d), ν denotes the unit normal to Γ pointed outward. Recently there has been a keen resurgence of interest (e.g. [L-L.1], [R.3] and references cited therein) in the theory of plate equations, of which the Euler-Bernoulli equation (1.1a) is a canonical model, presumably stimulated by two main sources:

i) renewed studies in the dynamics, feasibility, and implementation of so-called large scale flexible structures envisioned to be employed in space;

ii) recent mathematical advances in regularity theory of second order mixed hyperbolic problems (canonically, the wave equation of both Dirichlet Type [L.8] [L-T.8] [L-T.9] [L-L-T.1] and of Neumann type [L-T.8], [L-T.9], [L-T.10], [L-T.13], [L-T.14], [L-T.15], [S.1]) with L^2 -boundary data.

In either case, a prime thrust of motivation has come from dynamical control studies, at either an engineering or a theoretical level. With reference to the specific problem (1.1), we cite, [L.7], [L-T.1], [L-T.2] for optimal regularity theory and exact controllability theory with respect to classes of interest for the initial data $[\,w_0, w_1\,]$ and for the boundary data $\{g_1, g_2\}$, which markedly improved upon prior literature [L-M]. Our interest in the present work is on the problem of boundary feedback uniform stabilization

for the dynamics (1.1) by explicit feedback operators, to be more properly defined below. Our results are fully consistent with the corresponding exact controllability results [L-T.1], [L-T.2], not only with respect to the function spaces for $\{g_1,g_2\}$ and $[w,w_t]$ as mentioned above, but also with respect to the lack of geometrical conditions on Ω when both g_1 and g_2 are active, or else with respect to the presence of similar geometrical conditions on Ω when only g_1 is active and g_2 is taken $g_2 = 0$. This consistency is most desirable, and indeed has to be expected, in view of the known relationship between exact controllability and uniform stabilization for time reversible dynamics such as (1.1) [R.1] [R.2]. We note, in passing, that uniform stabilization of problem (1.1) by means of a feedback operator (acting on $[w,w_t]$) which is defined in terms of the algebraic Riccati operator (which arises in the study of the optimal quadratic cost problem on $(0,\infty)$) was already achieved in the abstract treatment of [F-L-T.1]. Mathematically the present work is guided by and partially rests upon techniques developed in two main sources:

(i) the studies of exact controllability [L-T.1], [L-T.2] for problem (1.1);

(ii) the study of uniform stabilization of the wave equation with boundary feedback in the Dirichlet boundary conditions [L-T.4] and in the Neumann boundary conditions [T.1].

Of course these studies have to be seen in the context of recent investigations including:

a) uniform stabilization of the wave equation with feedback in the Neumann boundary conditions [C.1], [C.2], [L.4], [L.5], [L-T.4. sect 4];

b) regularity theory for hyperbolic equations in [L.8], [L-T.8], [L-T.9], [L-L-T.1], as well as corresponding exact controllability theory [L.6], [L-T.12], [H.1], [T.2], [T.3];

c) exact controllability results for Euler-Bernoulli equations with different boundary conditions [L.6], [L.7], [L-T.6], [L-L.1]; and, finally,

d) corresponding optimal quadratic cost problems [D-L-T.1], [L-T.11], [F-L-T.1].

A point of view which we stress is the following: we choose g_1 in 'open loop' form to be in $L^2((0,T);L^2(\Gamma))$, $T<\infty$. This determines that the corresponding solution of problem (1.1) with $w_0 = w_1 = g_2 = 0$ satisfies $[w,w_t] \in C([0,T];\mathcal{Z})$, where $\mathcal{Z}$ is the space identified in (1.7) below. This is an optimal regularity result [L.7], [L-T.2]. Next, we choose g_2 such that the corresponding solution of

(1.1) with $g_1 = w_0 = w_1 = 0$ also produces $[w, w_t] \in C([0, T]; \mathcal{Z})$, again as an optimal regularity result. This leads to $[w, w_t] \in \mathcal{Z}$ and $g_2 \in L^2((0,T); H^{-1}(\Gamma))$ [L.7], [L-T.1] [L-T.2].

In other words, only one choice is made, that $g_1 \in L^2((0,T); L^2(\Gamma))$; then, we work with data and solutions in the corresponding optimal spaces. Our solution of the uniform stabilization problem below is fully consistent with these 'open loop' considerations: uniform stabilization will be achieved in the space $\mathcal{Z}$ with controls in feedback form $g_1 \in L^2((0,\infty); L^2(\Gamma))$ and $g_2 \in L^2((0,\infty); H^{-1}(\Gamma))$, (see Theorem 1.2 below). However, the present thesis will treat only the case $g_1 \in L^2((0,\infty); L^2(\Gamma))$ in feedback form and $g_2 = 0$, leaving the more general case to our successive effort [B-T.1].

We finally point out that other uniform stabilization problems for plate equations have been/are being considered following Lagnese-Lions recent monograph [L-L.1] on plates [L.1], [L.3]. In these works, however, different boundary conditions occur, typically of higher order, e.g., Δw and $\frac{\partial(\Delta w)}{\partial \nu}$. As a consequence, uniform stabilization results are sought and obtained in higher topologies on Ω (so called corresponding 'energy spaces'). The problem considered in this thesis with boundary conditions as in (1.1c), (1.1d) of lower order have a natural (and optimal, in fact) setting in very low topologies on Ω; see the optimal space below. This produces additional mathematical difficulties. To overcome these obstacles, it will be necessary to introduce a new variable, the variable p below in (3.3), which lifts the topologies on Ω to the level where the multiplier techniques which were successfully used in [L-T.1], [L-T.2] for the corresponding exact controllability problem are applicable. A similar lifting was employed in the case of the uniform stabilization problem for wave equations with Dirichlet feedback in the (low, but optimal) topology $L^2(\Omega) \times H^{-1}(\Omega)$ [L-T.4].

1.2 Formulation of the Uniform Stabilization Problem and Main Statements

Throughout the paper we let $< \ , \ >_\Omega$ denote the $L^2(\Omega)$-inner product with associated norm $\| \ \|_\Omega$, and $< \ , \ >_\Gamma$ denote the $L^2(\Gamma)$-inner product with associated norm $\| \ \|_\Gamma$. In addition, $\mathbf{L}(\Omega)$ denotes the Hilbert space of all bounded linear operators on $L^2(\Omega)$ and $\mathbf{L}(\Gamma,\Omega)$ denotes the Hilbert space of all bounded linear operators from $L^2(\Gamma)$ into $L^2(\Omega)$. Finally, $\mathbf{L}(X)$ will denote the Banach space of all bounded linear operators on X.

We begin by letting $A: \mathcal{D}(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be the positive, self-adjoint operator defined by

$$Af \doteq \Delta^2 f; \quad \mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega) \tag{1.2}$$

With the operator A defined as such, it then follows that [L-T.2, Appendix C]

$$\mathcal{D}(A^{1/4}) \simeq H_0^1(\Omega) \doteq \left\{ f \in H^1(\Omega) \,:\, f\Big|_\Gamma = 0 \right\} \tag{1.3}$$

$$\mathcal{D}(A^{3/4}) \simeq V \doteq \left\{ f \in H^3(\Omega) \,:\, f\Big|_\Gamma = \frac{\partial f}{\partial \nu}\Big|_\Gamma = 0 \right\} \tag{1.4}$$

where we use $\simeq$ to denote norm equivalence.
Thus, for $f \in \mathcal{D}(A^{1/4})$

$$\| f \|_{\mathcal{D}(A^{1/4})} \doteq \| A^{1/4} f \|_\Omega \simeq \| f \|_{H_0^1(\Omega)} \simeq \left\{ \int_\Omega |\nabla f|^2 d\Omega \right\}^{1/2} \tag{1.5}$$

where the last equivalence follows by Poincare inequality.
Similarly for $f \in \mathcal{D}(A^{3/4})$

$$\| f \|_{\mathcal{D}(A^{3/4})} \doteq \| A^{3/4} f \|_\Omega \simeq \left\{ \int_\Omega |\nabla(\Delta f|^2 d\Omega \right\}^{1/2} \tag{1.6}$$

As suggested by [L-T.1], [L-T.2], our optimal space in which to study stabilization will be,

$$Z = H^{-1}(\Omega) \times V' = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]' \tag{1.7}$$

where $'$ denotes duality with respect to the $L^2(\Omega)$ topology. Next, let $g_1 = g_2 = 0$ in (1.1). Then, the corresponding evolution of (1.1) is governed by the operator

$$\mathbf{A}_0 = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \tag{1.8}$$

which generates a strongly continuous unitary group (on the space $\mathcal{D}(A^{1/2}) \times L^2(\Omega)$ with domain $\mathcal{D}(\mathbf{A}_0) = \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and hence) on the space $\mathbf{Z}$ of our interest with domain $\mathcal{D}(\mathbf{A}_0) = \mathcal{D}(A^{1/4}) \times [\mathcal{D}(A^{1/4})]' = H_0^1(\Omega) \times H^{-1}(\Omega)$. We denote this unitary group by $e^{\mathbf{A}_0 t}$. Thus, the free solutions of (1.1) with $g_1 = g_2 = 0$ are norm-preserving in $\mathbf{Z}$:

$$\| [\, w(t), w_t(t) \,] \|_Z \equiv \| e^{\mathbf{A}_0 t} [\, w_0, w_1 \,] \|_Z = \| [\, w_0, w_1 \,] \|_Z, \text{ for all } t \in \mathbb{R}$$

With this well-known result at hand, we can state the aim of the paper. Motivated by and consistent with the function spaces in the optimal regularity and exact controllability theory of (1.1) [L.7], [L-T.1], [L-T.2], we shall study the question of existence and construction of explicit boundary feedback operators F_1 and F_2 based on the 'velocity' w_t

$$F_1(w_t) \in L^2((0,\infty)\ ; L^2(\Gamma)) \tag{1.9}$$

$$F_2(w_t) \in L^2((0,\infty)\ ; H^{-1}(\Gamma)) \tag{1.10}$$

such that the boundary feedback functions

$$g_1 = F_1(w_t), \qquad g_2 = F_2(w_t) \tag{1.11}$$

once inserted in (1.1 c-d) produce a (feedback) C_0-semigroup $e^{\mathbf{A}t}$ which is exponentially stable in the uniform operator norm $\mathbf{L}(\mathbf{Z})$ of the space $\mathbf{Z}$ in (1.7): namely, there exist constants $M \geq 1, \delta > 0$ such that for all $t \geq 0$

$$\| e^{\mathbf{A}t} \|_{\mathbf{L}(\mathbf{Z})} \leq Me^{-\delta t} \tag{U.S.}$$

We then say that such operators F_1 and F_2 uniformly (exponentially) stabilize the original norm-preserving (conservative) dynamics (1.1) with $g_1 = g_2 = 0$.

A weaker concept is that of 'strong stabilization', by which we seek operators F_1 and F_2 as in (1.11) such that once inserted in (1.1 c-d) produce a (feedback) C_0-semigroup $e^{\mathbf{A}t}$ on $\mathbf{Z}$ which decays strongly to zero.

$$\| e^{\mathbf{A}t} \|_{\mathbf{Z}} \to 0 \quad \text{as } t \to \infty, \quad \text{for all } z \in \mathbf{Z} \tag{S.S}$$

We note that for compact semigroups such as those arising in parabolic equations on a bounded domain Ω, the concept of stability in the strong topology (even weak topology) is equivalent to the concept of stability in the uniform topology. However, for second order hyperbolic problems and plate problems such as (1.1), the two concepts of strong and uniform stability are distinct. For recent optimal results on the lack of uniform stabilization, see [T.5], [T.6].

Choice of Operators F_1 and F_2

It is justified in Chapter 2 in the case of F_1 and in [B-T.1] in the general case that the following choices of F_1 and F_2

$$g_1 = F_1(w_t) = -k_1(x)G_1^* A^{-1/2} w_t = -k_1(x)G_1^* AA^{-3/2} w_t = -k_1(x)\frac{\partial \Delta(A^{-3/2})w_t}{\partial \nu}\Big|_\Sigma \tag{1.12}$$

$$g_2 = F_2(w_t) = -k_2(x)\Lambda^2 G_2^* AA^{-3/2} w_t = k_2(x)\Lambda^2 \Delta(A^{-3/2} w_t)\Big|_\Sigma \tag{1.13}$$

provide reasonable candidates for the uniform stabilization problem of (1.1), in the sense that the closed loop feedback dynamics with (1.12) and (1.13) inserted in (1.1c) and (1.1d), respectively, is well-posed in the semigroup sense in $\mathbf{Z}$ and the $\mathbf{Z}$-norm of all of its solutions originating in $\mathbf{Z}$ decreases as $t \to +\infty$ (this, however, does not say that such $\mathbf{Z}$-norms decreases to zero as $t \to +\infty$, (strong stabilization), let alone in the uniform norm of $\mathbf{L}(\mathbf{Z})$). To show this conclusion will be our major task in Chapter 3, at least in the case where $g_2 \equiv 0$ at the outset. In (1.12), (1.13) we have that:

a) $k_i(x)$ = smooth functions on Γ, $k_i(x) > 0$; (1.14)

b) Λ : (onto) isomorphism $H^s(\Gamma) \to H^{s-1}(\Gamma)$, self-adjoint on $L^2(\Gamma)$ (1.15)

so that

$$\| \Lambda g \|_{L^2(\Gamma)} = \| g \|_{H^1(\Gamma)} \approx \left\{ \int_\Gamma |\nabla_\sigma g|^2 d\Gamma \right\}^{1/2} \tag{1.16}$$

where ∇_δ denotes the tangential gradient on Γ;

c) The operators G_i^* are the adjoints, in the sense that

$$< G_i g \, , \, z >_\Omega = < g \, , \, G_i^* z >_\Gamma, \; g \in L^2(\Gamma), z \in L^2(\Omega) \tag{1.17}$$

of the operators G_i defined by:

$$G_1 g = x \quad \text{if and only if} \begin{cases} \Delta^2 x = 0 \quad \text{in } \Omega & (a) \\ x\big|_\Gamma = g_1 \quad \text{on } \Gamma & (b) \\ \dfrac{\partial x}{\partial \nu}\Big|_\Gamma = 0 \quad \text{on } \Gamma & (c) \end{cases} \tag{1.18}$$

$$G_2 g = y \quad \text{if and only if} \begin{cases} \Delta^2 y = 0 \quad \text{in } \Omega & (a) \\ y\big|_\Gamma = 0 \quad \text{on } \Gamma & (b) \\ \dfrac{\partial y}{\partial \nu}\Big|_\Gamma = g_2 \quad \text{on } \Gamma & (c) \end{cases} \tag{1.19}$$

Elliptic theory [L-M, Vol. 1] gives for any $s \in \mathbb{R}$

$$G_1 : \text{continuous } H^s(\Gamma) \to H^{s+1/2}(\Omega) \tag{1.20}$$

$$G_2 : \text{continuous } H^s(\Gamma) \to H^{s+3/2}(\Omega) \tag{1.21}$$

Moreover, by Green's theorem it is proved that [L-T.2, Lemma 2.0 and Lemma 4.0, respectively]

$$G_1^* A f = \frac{\partial(\Delta f)}{\partial \nu}\Big|_\Gamma, \quad f \in \mathcal{D}(A) \tag{1.22}$$

$$G_2^* A f = -(\Delta f)\Big|_\Gamma, \quad f \in \mathcal{D}(A) \tag{1.23}$$

Identities (1.22), (1.23) are used in the last step of (1.12), (1.13), respectively. Thus, the resulting candidate feedback system, whose stability properties in $\mathcal{Z}$ we shall investigate is,

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } (0,\infty)\times\Omega = Q \quad (a) \\ w(0,x) = w_0(x)\,;\, w_t(0,x) = w_1(x) & \text{in } \Omega \quad (b) \\ w\Big|_\Sigma = -k_1(x)\dfrac{\partial\Delta(A^{-3/2}w_t)}{\partial\nu}\Big|_\Sigma & \text{on } (0,\infty)\times\Gamma = \Sigma \quad (c) \\ \dfrac{\partial w}{\partial\nu}\Big|_\Sigma = k_2(x)\Lambda^2\Delta(A^{-3/2}w_t)\Big|_\Sigma & \text{on } \Sigma \quad (d) \end{cases} \tag{1.24}$$

Using the techniques of [T.4], problem (1.24) can be re-written more conveniently in abstract form as

$$\frac{d}{dt}\begin{vmatrix} w \\ \dot{w} \end{vmatrix} = \mathbf{A}\begin{vmatrix} w \\ \dot{w} \end{vmatrix} \quad on\ \mathcal{Z} \tag{1.25}$$

$$\mathbf{A} = \begin{vmatrix} 0 & I \\ -A & -A\,[k_1 G_1 G_1^* A^{-1/2} + k_2 G_2 \Lambda^2 G_2^* A^{-1/2}] \end{vmatrix} \tag{1.26a}$$

$$\mathcal{D}(\mathbf{A}) \doteq \{\, y \in \mathcal{Z} : \mathbf{A}y \in \mathcal{Z} \,\} \tag{1.26b}$$

A more explicit description of $\mathcal{D}(\mathbf{A})$ will be given below. Our main results are as follows.

Theorem 1.1

(i) Well-posedness on $\mathcal{Z}$

The operator $\mathbf{A}$ in (1.26) is dissipative on $\mathcal{Z} = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]'$, see (1.7), and satisfies here: range $(\lambda I - \mathbf{A}) = \mathcal{Z}$ for $\lambda > 0$. Thus, by Lumer-Phillips theorem [P], $\mathbf{A}$ generates a strongly continuous contraction semigroup $e^{\mathbf{A}t}$ on $\mathcal{Z}$, and the resolvent operator $R(\lambda\,;\mathbf{A})$ is compact on $\mathcal{Z}$, for $\mathrm{Re}\lambda > 0$.

Moreover $0 \in \rho(\mathbf{A})$, the resolvent set of $\mathbf{A}$.

(ii) Boundedness of feedback operators

For $[\, w_0, w_1 \,] \in \mathbf{Z}$, we have

$$k_1 G_1^* A^{-1/2} w_t = k_1 \frac{\partial \Delta (A^{-3/2} w_t)}{\partial \nu} \in L^2(\,(0,\infty)\,; L^2(\Gamma)\,) \tag{1.27}$$

$$k_2 \Lambda^2 G_2^* A^{-1/2} w_t = k_2 \Lambda^2 \Delta (A^{-3/2} w_t) \in L^2(\,(0,\infty)\,; H^{-1}(\Gamma)\,) \tag{1.28}$$

more precisely

$$\int_0^\infty \| \, k_1 G_1^* A^{-1/2} w_t \, \|_\Gamma^2 dt \le \| \, [\, w_0, w_1 \,] \, \|_{\mathbf{Z}}^2 \tag{1.29}$$

$$\int_0^\infty \| \, k_2 \Lambda^2 G_2^* A^{-1/2} w_t \, \|_\Gamma^2 dt \le \| \, [\, w_0, w_1 \,] \, \|_{\mathbf{Z}}^2 \tag{1.30}$$

The proof of Theorem 1.1 will be given in Chapter 2, in the case of $g_2 \equiv 0$, and in [B-T.1] in the general case.

Theorem 1.2 (Uniform stabilization on $\mathbf{Z}$ with both feedback operators in the absence of geometrical conditions on Ω)

The following property holds for the feedback problem (1.24), or (1.25), (1.26): there are constants $M \ge 1$ and $\delta > 0$ such that for all $t \ge 0$

$$\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \|_{\mathbf{Z}} = \| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|_{\mathbf{Z}} \le M e^{-\delta t} \| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|_{\mathbf{Z}} \tag{1.31}$$

Theorem 1.3 (Uniform stabilization on $\mathbf{Z}$ with only the first feedback operator g_1 and $g_2 \equiv 0$, in the presence of geometrical conditions on Ω).

Consider the feedback problem (1.1) with g_1 given by (1.12) while $g_2 \equiv 0$. Then there is a constant $\delta > 0$ such that the uniform decay (1.31) holds true, provided Ω satisfies the following geometrical condition (radial vector field assumption):

there exists a point $x_0 \in I\!R^n$ such that $(x - x_0) \cdot \nu \ge \gamma > 0$ on Γ (1.32)

Theorem 1.3 is the main result proved in this thesis. It may be extended to more general domains Ω which satisfy a weaker geometrical condition than (1.32), expressed in terms of a more general vector field than the class of radial fields $(x - x_0)$. This will be done in [B-T.1]. Also Theorem 1.2--the general result

with two feedback operators and no geometrical condition--will likewise appear in [B-T.1]. Instead, in Chapter 2 of this thesis we shall also prove a strong stabilization result (Theorem 2.10) with g_1 as in (1.12), (and k_1=1) and $g_2 = 0$, under the same geometrical assumption (1.32). Though the strong stabilization result of Theorem 2.10 in Chapter 2 is implied by the uniform stabilization result of Theorem 1.3 in Chapter 3, we feel that its inclusion in this thesis is justified by the following considerations. It shows 'how far' the purely operator techniques--based on the operator model (1.25), (1.26) and the Nagy-Foias-Foguel decomposition for contraction semigroups--can be carried out along the lines of arguments first used for boundary control problems for second order hyperbolic equations in [L-T.16], [L-T.17], [L-T.4] and also in [T.1] for different feedback operators. A new obstacle arises, however, in the case of plate problems (fourth order in the space variable rather than second order in space as in hyperbolic problems) at the level of excluding the presence of eigen-values along the imaginary axis. This is Lemma 2.7, where only three homogeneous boundary conditions (2.31b-c-d) are in place for the 'eigen-problem' (2.31a), as opposed to the four homogeneous boundary conditions for a fourth order operator covered by standard theory. (In previous arguments for second order hyperbolic problems as in [L-T.16], [L-T.17], [L-T.4], [T.1], the corresponding 'eigen-problem' has two homogeneous boundary conditions for a second order operator and hence is covered by standard theory). A novelty is then that Lemma 2.7 uses a multiplier technique proof, which is supported by the exact controllability problem [L-T.1], [L-T.2], as applied however to the (stationary) elliptic problem this time. It is the multiplier technique that requires the geometrical condition (1.32) in the proof of Lemma 2.7. Lemma 2.7 appears to be new in elliptic theory. This fact alone would justify its inclusion here, even though Lemma 2.7 plays only a secondary role in the problem of stabilization considered here, as it leads only to strong stabilization (Theorem 2.10). It is through the more elaborate and lengthier arguments of Chapter 3--which are necessitated by use of the same multiplier applied to the feedback dynamics (1.24)--that we will eventually obtain the uniform stabilization result of Theorem 1.3.

CHAPTER 2
WELL-POSEDNESS AND STRONG STABILIZATION

2.1 Preliminaries, Choice of Stabilizing Feedback

Let Ω be an open bounded domain in $I\!R^n$, $n \geq 2$ with sufficiently smooth boundary Γ. Consider the nonhomogeneous problem in the solution $w(t,x)$:

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q = (0,\infty)\times\Omega \quad (a) \\ w(0,\cdot) = w_0 \quad w_t(0,\cdot) = w_1 & \text{in } \Omega \quad (b) \\ w\Big|_{\Sigma} = g_1 \in L^2(\Sigma) = L^2((0,\infty); L^2(\Gamma)) & \text{on } \Sigma = (0,\infty)\times\Gamma \quad (c) \\ \dfrac{\partial w}{\partial \nu}\Big|_{\Sigma} = g_2 = 0 & \text{on } \Sigma \quad (d) \end{cases} \tag{2.1}$$

The goal of this chapter is to obtain strong stabilization of the system (2.1) via a closed-loop feedback g_1 based on the velocity w_t. However, the optimal function space in which to work (a cross product space for position and velocity) is obtained from the exact controllability result to be summarized below.

First, we define the positive, self-adjoint operator A: $I\!D(A) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$Af \doteq \Delta^2 f \tag{2.2a}$$

$$I\!D(A) = \left\{ f \in L^2(\Omega) : \Delta^2 f \in L^2(\Omega),\ f\Big|_{\Gamma} = \frac{\partial f}{\partial \nu}\Big|_{\Gamma} = 0 \right\} \equiv H^4(\Omega) \cap H_0^2(\Omega) \tag{2.2b}$$

Since Ω is bounded in $I\!R^n$, then A has compact resolvent $R(\lambda; A)$. Also, if $Af = 0$ for $f \in I\!D(A)$ then by (A.0) and (2.1c,d) we have

$$\begin{aligned} 0 = < Af,\ f >_{\Omega} &= < \Delta(\Delta f),\ f >_{\Omega} \\ &= < \Delta f,\ \Delta f >_{\Omega} + < \frac{\partial(\Delta f)}{\partial \nu},\ f >_{\Gamma} - < \Delta f,\ \frac{\partial f}{\partial \nu} >_{\Gamma} \\ &= < \Delta f,\ \Delta f >_{\Omega} = \| \Delta f \|_{\Omega}^2 \end{aligned}$$

This implies $f = 0$, so therefore,

$$A^{-1} \in \mathbf{L}(\Omega) \tag{2.3}$$

Next, we let

$$V = \left\{ f \in H^3(\Omega) : f\Big|_\Gamma = \frac{\partial f}{\partial \nu}\Big|_\Gamma = 0 \right\} \tag{2.4a}$$

and consider the space $\mathbf{Z} = H^{-1}(\Omega) \times V'$. As shown in [L-T.2], $\mathbf{Z}$ can be characterized by using equivalent norms as

$$\mathbf{Z} = [\mathbb{D}(A^{1/4})]' \times [\mathbb{D}(A^{3/4})]' \tag{2.4b}$$

where $'$ denotes duality with respect to the $L^2(\Omega)$-topology.

The norms on these spaces are given by

(2.5)

$$\| x \|_{\mathbb{D}(A^\alpha)} \doteq \| A^\alpha x \|_\Omega ; \quad \| x \|_{[\mathbb{D}(A^\beta)]'} \doteq \| A^{-\beta} x \|_\Omega \quad \alpha, \beta \geq 0 \tag{2.5}$$

Below we state the regularity result as well as the exact controllability result.

Theorem 2.0 (Regularity) [L.7], [L-T.2]

Consider the problem (2.1) subject to $[w_0, w_1] \in \mathbf{Z}$, $g_1 \in L^2((0,T); L^2(\Gamma))$, $g_2 \in L^2((0,T); H^{-1}(\Gamma))$. Then the map $\{w_0, w_1, g_1, g_2\} \to [w(t), w_t(t)] \in C([0,T]; \mathbf{Z})$ is continuous for any $0 < T < \infty$. □

Theorem 2.1 (Exact Controllability) [L-T.2]

(i) Assume there exists a point $x_0 \in \mathbb{R}^n$ such that $(x - x_0) \cdot \nu \geq \gamma > 0$ on Γ where ν is the unit outward normal vector. Let $0 < T < \infty$ be arbitrary. If $[w_0, w_1] \in \mathbf{Z}$ arbitrary, then there exists a suitable control function $g_1 \in L^2((0,T); L^2(\Gamma))$, such that the corresponding solution of (2.1) with $g_2 \equiv 0$ satisfies $w(T,\cdot) = w_t(T,\cdot) \equiv 0$ and in addition $[w, w_t] \in C([0,T]; \mathbf{Z})$

(ii) The same conclusion holds true without geometrical conditions if g_2 is taken within the class of $L^2((0,T); H^{-1}(\Gamma))$ controls. □

By time reversibility, we see that at any finite T the totality of all solution points $\{w(T), w_t(T)\}$ of problem (1.1) with $w_0 = w_1 = 0$ fills all of the space $\mathbf{Z}$ when either g_1 runs over all of $L^2((0,T); L^2(\Gamma))$

and $g_2 \equiv 0$ under geometrical conditions on Ω, or else when $\{g_1, g_2\}$ runs over all of $L^2((0,T);L^2(\Gamma)) \times L^2((0,T);H^{-1}(\Gamma))$ without geometrical conditions. Therefore, since the space of exact controllability is the space of maximal regularity, we seek stabilization in exactly this space $\mathcal{Z}$.

We define the "energy" $E(t)$ for the dynamics (2.1) over the space $\mathcal{Z} = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]'$ by

$$E(t) \doteq \| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \|_{\mathcal{Z}}^2 \doteq \| w(t) \|_{[\mathcal{D}(A^{1/4})]'}^2 \times \| w_t(t) \|_{[\mathcal{D}(A^{3/4})]'}^2$$
$$\doteq \| A^{-1/4} w(t) \|_{\Omega}^2 + \| A^{-3/4} w_t(t) \|_{\Omega}^2 \tag{2.6}$$

Next we seek a candidate g_1 which at least produces $\frac{dE}{dt} \leq 0$, i.e., energy "decrease." This does not, however, guarantee $\lim_{t \to \infty} E(t) = 0$ (which is precisely strong stability of (2.1)), let alone uniform stability.

Remark 2.2

Below we shall show well-posedness in $\mathcal{Z}$, with $g_2 = 0$. Then since $w_t \in [\mathcal{D}(A^{3/4})]'$, it follows that $A^{-3/2} w_t = A^{-3/4} A^{-3/4} w_t \in \mathcal{D}(A^{3/4}) \equiv V$. Therefore, $A^{-3/2} w_t$ satisfies the required boundary conditions.

$$A^{-3/2} w_t \Big|_{\Gamma} = \frac{\partial}{\partial \nu} (A^{-3/2} w_t) \Big|_{\Gamma} \equiv 0 \tag{2.7}$$

By writing $E(t) = < A^{-1/4} w \, , \, A^{-1/4} w >_{\Omega} + < A^{-3/4} w_t \, , \, A^{-3/4} w_t >_{\Omega}$ and differentiating with respect to t we have that

$$\frac{1}{2} \frac{dE}{dt} = < A^{-1/4} w \, , \, A^{-1/4} w_t >_{\Omega} + < A^{-3/4} w_{tt} \, , \, A^{-3/4} w_t >_{\Omega}$$

by (2.1a)

$= < w \, , \, A^{-1/2} w_t >_{\Omega} - < Aw \, , \, A^{-3/2} w_t >_{\Omega} = < w \, , \, A^{-1/2} w_t >_{\Omega} - < \Delta(\Delta w) \, , \, A^{-3/2} w_t >_{\Omega}$

by (A.O)

$$= < w \, , \, A^{-1/2} w_t >_{\Omega} - \left[< \frac{\partial}{\partial \nu} (\Delta w) \, , \, A^{-3/2} w_t >_{\Gamma} - < \Delta w \, , \, \frac{\partial}{\partial \nu} (A^{-3/2} w_t) >_{\Gamma} + < \Delta w \, , \, \Delta (A^{-3/2} w_t) >_{\Omega} \right]$$

by (2.7)

$= < w \, , \, A^{-1/2} w_t >_{\Omega} - < \Delta w \, , \, \Delta (A^{-3/2} w_t) >_{\Omega}$

by (A.0)

$$= < w , A^{-1/2} w_t >_\Omega - \left[< \frac{\partial w}{\partial \nu} , \Delta(A^{-3/2} w_t) >_\Gamma - < w , \frac{\partial}{\partial \nu}[\Delta(A^{-3/2} w_t)] >_\Gamma + < w , \Delta^2 (A^{-3/2} w_t) >_\Omega \right]$$

$$= < w , A^{-1/2} w_t >_\Omega + < w , \frac{\partial}{\partial \nu}[\Delta(A^{-3/2} w_t)] >_\Gamma - < w , A^{-1/2} w_t >_\Omega = < w , \frac{\partial}{\partial \nu}[\Delta(A^{-3/2} w_t)] >_\Gamma$$

Therefore, by selecting the simplest choice

$$w\Big|_\Sigma = g_1 = -\frac{\partial}{\partial \nu}[\Delta(A^{-3/2} w_t)] \tag{2.8}$$

We obtain $\frac{dE}{dt} = -2 \,||\, g_1 \,||^2_\Gamma \le 0$, our desired energy decrease. □

Next we will show how our feedback can be expressed in terms of an operator (Green map) which acts from boundary Γ to interior Ω. Following [L-T.1],[L-T.2] we define $G_1 : L^2(\Gamma) \to L^2(\Omega)$ by

$$G_1 g = y \text{ if and only if } \begin{cases} Ay = 0 & \text{in } \Omega \quad (a) \\ y\Big|_\Gamma = g & \text{on } \Gamma \quad (b) \\ \frac{\partial y}{\partial \nu}\Big|_\Gamma = 0 & \text{on } \Gamma \quad (c) \end{cases} \tag{2.9}$$

We quote the following Lemma which will be used below.

Lemma 2.3 [L-T.2]

Let $G_1^* : L^2(\Omega) \to L^2(\Gamma)$ denote the continuous operator defined by $< G_1 g , v >_\Omega = < g , G_1^* v >_\Gamma$, $g \in L^2(\Gamma)$, $v \in L^2(\Omega)$, i.e., G_1^* is the adjoint of G_1. Then

$$G_1^* A f = \frac{\partial}{\partial \nu}(\Delta f)\Big|_\Gamma \quad \text{for } f \in \mathcal{D}(A) \tag{2.10}$$

Now using (2.8) and (2.10) we see that

$$w\Big|_\Sigma = g_1 = -\frac{\partial}{\partial \nu}[\Delta(A^{-3/2} w_t)] = -G_1^* A (A^{-3/2} w_t) = -G_1^* A^{-1/2} w_t \tag{2.11}$$

Using elliptic theory [L-M, Vol I, p. 188] we have that for any s real

$$G_1 : \text{continuous } H^s(\Gamma) \to H^{s+1/2}(\Omega) \tag{2.12a}$$

and in particular for $s = 0$

$$G_1 : \text{continuous } L^2(\Gamma) \to H^{1/2}(\Omega) \tag{2.12b}$$

We also have that by duality on (2.12a) with $s = -3/2$ that

$$G_1^* : \text{ continuous } H_0^1(\Omega) \to H^{3/2}(\Gamma) \tag{2.13}$$

so that (2.12a), (2.13) imply

$$G_1 G_1^* : \text{ continuous } \mathcal{D}(A^{1/4}) = H_0^1(\Omega) \to H^2(\Omega) \tag{2.14}$$

to be used below (2.17b) in the description of the domain of the feedback generator.

2.2 Well-Posedness and Semigroup Generation

First we want to introduce an abstract operator model for problem (2.1). According to [T.1], [T.4], problem (2.1) with $g_2 = 0$ admits the following abstract versions:

as a second order equation

$$\ddot{w} = -A\,[\,w - G_1 g_1\,] = -A\,[\,w + G_1 G_1^* A^{-1/2}\dot{w}\,] \tag{2.15a}$$

or else as a first order system

$$\frac{d}{dt}\begin{vmatrix} w \\ \dot{w} \end{vmatrix} = \mathbf{A}\begin{vmatrix} w \\ \dot{w} \end{vmatrix} \;;\quad [\,w\,,\ \dot{w}\,] \in \mathcal{Z} = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]' \tag{2.15b}$$

$$\text{where } \mathbf{A} = \begin{vmatrix} 0 & I \\ -A & -AG_1 G_1^* A^{-1/2} \end{vmatrix} \tag{2.16}$$

More explicitly if $y \in \mathcal{D}(\mathbf{A})$ then we can write

$$\mathbf{A}y = \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \begin{vmatrix} y_1 + G_1 G_1^* A^{-1/2} y_2 \\ y_2 \end{vmatrix} \tag{2.17a}$$

Thus, $\mathcal{D}(\mathbf{A}) = \Big\{\, [\,y_1\,,\ y_2\,] \in \mathcal{Z} : y_2 \in [\mathcal{D}(A^{1/4})]'$ and $-A\,[\,y_1 + G_1 G_1^* A^{-1/2} y_2\,] \in [\mathcal{D}(A^{3/4})]'$, i.e.,

$$y_1 + G_1 G_1^* A^{-1/2} y_2 \in \mathcal{D}(A^{1/4}) \equiv H_0^1(\Omega) \text{ which implies } y_1 \in H^1(\Omega) \Big\} \tag{2.17b}$$

The operator $\mathbf{A}$ defined above is our candidate to be the generator of a feedback semigroup. The first step in this direction is the following Lemma.

Lemma 2.5

The operator $\mathbf{A}$ is dissipative on $\mathcal{Z} = [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]'$.

Proof

Let $z \in \mathcal{Z}$, then using below the skew-adjointness of $\begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix}$ we have for $z \in \mathcal{D}(\mathbf{A})$

$$Re < \mathbf{A}z\,,\ z >_{\mathcal{Z}} = Re < \begin{vmatrix} 0 & I \\ -A & 0 \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix},\ \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} >_{\mathcal{Z}}$$

$$+ Re < \begin{vmatrix} 0 & 0 \\ 0 & -AG_1G_1^*A^{-1/2} \end{vmatrix} \begin{vmatrix} z_1 \\ z_2 \end{vmatrix},\ \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} >_{\mathcal{Z}}$$

$$= 0 - < AG_1G_1^*A^{-1/2}z_2\,,\ z_2 >_{[\mathcal{D}(A^{3/4})]'} = < A^{-3/4}AG_1G_1^*A^{-1/2}z_2\,,\ A^{-3/4}z_2 >_{\Omega}$$

$$= - \| G_1^*A^{-1/2}z_2 \|_{\Gamma}^2 \le 0 \text{ and dissipativity holds. } \square$$

The above proof is a reformulation of our argument below (2.7). Now we come to our result on semigroup generation.

Theorem 2.6

(i) The dissipative operator $\mathbf{A}$ in (2.16) also satisfies range $(\lambda I - \mathbf{A}) \equiv \mathcal{Z}$ on $\mathcal{Z}$ for $\lambda > 0$.

Thus, by the Lumer-Phillips Theorem [P], $\mathbf{A}$ generates a C_0-semigroup of contractions $e^{\mathbf{A}t}$ on $\mathcal{Z}$, $t \ge 0$ and the solution of (2.1),(2.11) is given by

$$\begin{vmatrix} w(t\,,\ w_0\,,\ w_1) \\ w_t(t\,,\ w_0\,,\ w_1) \end{vmatrix} = e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \quad \text{all } t \ge 0\,, \quad [w_0\,,\ w_1] \in \mathcal{Z} \tag{2.18a}$$

and in fact

$$\| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|_{\mathcal{Z}}^2 = E(t) = \int_{\Omega} \left\{ |A^{-1/4}w|^2 + |A^{-3/4}w_t|^2 \right\} d\Omega \tag{2.18b}$$

(ii) The resolvent operator $R(\lambda\,;\mathbf{A})$ of $\mathbf{A}$ is given by

$$R(\lambda\,;\mathbf{A}) = \begin{vmatrix} \dfrac{I - V(\lambda)^{-1}}{\lambda} & V(\lambda)^{-1}A^{-1} \\ -V(\lambda)^{-1} & \lambda V(\lambda)^{-1}A^{-1} \end{vmatrix} \tag{2.19a}$$

$$\text{where } V(\lambda) = [I + \lambda G_1G_1^*A^{-1/2} + \lambda^2 A^{-1}] \tag{2.19b}$$

at least for all λ satisfying $Re\lambda > 0$. Moreover $\lambda = 0$ belongs to the resolvent set of $\mathbf{A}$ and $R(\lambda\,;\mathbf{A})$ is compact on $\mathcal{Z}$.

(iii) If the domain Ω is such that there exists a point $x_0 \in \mathbb{R}^n$ such that

$$(x-x_0)\cdot \nu \geq \gamma > 0 \quad \text{on } \Gamma \tag{2.20}$$

then $R(\lambda ; \mathbf{A})$ is well-defined and compact on $\mathcal{Z}$ also on the imaginary axis and hence, for all λ satisfying $Re\lambda \geq 0$. Thus, the spectrum (point) of $\mathbf{A}$ satisfies

$$\sigma(\mathbf{A}) \subset \{ \lambda : Re\lambda < 0 \} \tag{2.21}$$

Remark

A stronger result will follow below once we prove our uniform stabilization Theorem 1.3, that in fact $\sigma(\mathbf{A}) \subset \{ \lambda : Re\lambda \leq -\delta < 0 \}$. □

Proof of Theorem 2.6

Dissipativity of $\mathbf{A}$ on $\mathcal{Z}$ was already shown in Lemma 2.5. Next, fix $\lambda > 0$ and let $z \in \mathcal{Z}$ and we want to solve $(\lambda I - \mathbf{A})y = z$, i.e.,

$$\lambda y_1 - y_2 = z_1 \in [\mathcal{D}(A^{1/4})]' \tag{2.22a}$$

$$A(y_1 + G_1 G_1^* A^{-1/2} y_2) + \lambda y_2 = z_2 \in [\mathcal{D}(A^{3/4})]' \tag{2.22b}$$

for $y \in \mathcal{D}(\mathbf{A})$. We apply A^{-1} to (2.22b), multiply (2.22a) by λ and subtract to obtain:

$$V(\lambda) y_2 = \lambda A^{-1} z_2 - z_1 \in [\mathcal{D}(A^{1/4})]' \tag{2.23}$$

with $V(\lambda)$ defined in (2.19b).

We next note that $V(\lambda)$ is boundedly invertible on $[\mathcal{D}(A^{1/4})]'$ since equivalently $A^{-1/4} V(\lambda) A^{1/4} = I + \lambda A^{-1/4} G_1 G_1^* A^{-1/4} + \lambda^2 A^{-1}$ is boundedly invertible on $L^2(\Omega)$ (being self-adjoint, strictly positive on $L^2(\Omega)$) with inverse

$$A^{-1/4} V^{-1}(\lambda) A^{1/4} \in \mathbf{L}(\Omega) \tag{2.24}$$

Thus, from (2.23)

$$y_2 = V^{-1}(\lambda)\ (\lambda A^{-1} z_2 - z_1) \in [\mathcal{D}(A^{1/4})]' \tag{2.25}$$

which then inserted in (2.22a) yields

$$y_1 = \left[\frac{I - V^{-1}(\lambda)}{\lambda} \right] z_1 + V^{-1}(\lambda) A^{-1} z_2 \tag{2.26}$$

Then (2.19a) follows from (2.25) and (2.26). Note that from (2.22b) and (2.25) that

$$y_1 + G_1 G_1^* A^{-1/2} y_2 = A^{-1} z_2 - \lambda A^{-1} y_2 \in \mathcal{D}(A^{1/4}) \tag{2.27}$$

So that recalling (2.17b) we see that from (2.25) and (2.27) it is verified that $y \in \mathcal{D}(\mathbf{A})$. The compactness of $R(\lambda; \mathbf{A})$ on $\mathcal{Z}$ is readily seen from (2.19a) to be equivalent to compactness on $L^2(\Omega)$ of the following operators:

$$A^{-1/4}(I - V^{-1}(\lambda))A^{1/4} \tag{2.28a}$$

$$A^{-1/4} V^{-1}(\lambda) A^{-1/4} = A^{-1/4} V^{-1}(\lambda) A^{1/4} A^{-1/2} \tag{2.28b}$$

$$A^{-3/4} V^{-1}(\lambda) A^{1/4} = A^{-1/2} A^{-1/4} V^{-1}(\lambda) A^{1/4} \tag{2.28c}$$

$$A^{-3/4} V^{-1}(\lambda) A^{-1/4} = A^{-1/2} A^{-1/4} V^{-1}(\lambda) A^{1/4} A^{-1/2} \tag{2.28d}$$

First, compactness of the operators (2.28b-c-d) on $L^2(\Omega)$ is plain from (2.24) and $A^{-\alpha}$, $\alpha > 0$ being compact on $L^2(\Omega)$. For (2.28a) apply $V^{-1}(\lambda)$ on (2.19b) so that

$$I = V^{-1}(\lambda) + \lambda V^{-1}(\lambda) G_1 G_1^* A^{-1/2} + \lambda^2 V^{-1}(\lambda) A^{-1}$$

and then

$$\begin{aligned} A^{-1/4} [\, I - V^{-1}(\lambda) \,] A^{1/4} &= \lambda A^{-1/4} V^{-1}(\lambda) G_1 G_1^* A^{-1/4} + \lambda^2 A^{-1/4} V^{-1}(\lambda) A^{-3/4} \\ &= \lambda A^{-1/4} V^{-1}(\lambda) A^{1/4} A^{-1/4} G_1 G_1^* A^{-1/4} + \lambda^2 A^{-1/4} V^{-1}(\lambda) A^{1/4} A^{-1} \end{aligned}$$

which is compact on $L^2(\Omega)$ by (2.34) since $A^{-1/4} G_1 G_1^* A^{-1/4} \in \mathbf{L}(\Omega)$ □

To complete the proof of Theorem 2.6, we must show that $\sigma(\mathbf{A})$ does not contain any points on the imaginary axis (we already know that $\sigma(\mathbf{A})$ does not contain points in $\{ \mathrm{Re}\lambda > 0 \}$ since $\mathbf{A}$ is the generator of a contraction semigroup).

Thus, we need to show that

$$V(\lambda)^{-1} \in \mathbf{L}([\mathcal{D}(A^{1/4})]') \quad \text{for } \lambda = ir, \quad r \in \mathbb{R}, \ r \neq 0 \tag{2.29}$$

To this end let $x \in [\mathcal{D}(A^{1/4})]'$ and suppose $V(\lambda)x = 0$ for $\lambda = ir$. Then from (2.19b),

$$\begin{aligned} 0 &= < V(\lambda)x, \ x >_{[\mathcal{D}(A^{1/4})]'} \\ &= < x, \ x >_{[\mathcal{D}(A^{1/4})]'} + ir < G_1 G_1^* A^{-1/2} x, \ x >_{[\mathcal{D}(A^{1/4})]'} - r^2 < A^{-1} x, \ x >_{[\mathcal{D}(A^{1/4})]'} \\ &= < A^{-1/2} x, \ x >_{\Omega} + ir \, \| G_1^* A^{-1/2} x \|_{\Gamma}^2 - r^2 < A^{-3/2} x, \ x >_{\Omega} \end{aligned} \tag{2.30}$$

Since the middle term in (2.30) is purely imaginary we must have that via (2.10)

$$G_1^* A^{-1/2} x \Big|_{\Gamma} = G_1^* A [A^{-3/2} x] \Big|_{\Gamma} = \frac{\partial}{\partial \nu} [\Delta (A^{-3/2} x)]_{\Gamma} = 0 \tag{2.31}$$

Also, we have that by (2.30), $A^{-1/2}x = r^2 A^{-3/2}x$, i.e.,

$$Ax = r^2 x \tag{2.32}$$

which means that x must be an eigenvector of A say $x = e_n$ with eigenvalue r^2. Therefore, since $e_n \in \mathcal{D}(A)$ we have that it satisfies the two zero boundary conditions associated with $\mathcal{D}(A)$ (see (2.2b)), as well as (2.31). Therefore, the following Lemma will complete the proof of Theorem 2.6.

Lemma 2.7

Let $\lambda = r^2 > 0$ and suppose Ω satisfies the radial vector field assumption (2.20). Then the problem

$$\begin{cases} \Delta^2 \phi = \lambda \phi & \text{in } \Omega & (a) \\ \phi\big|_\Gamma = 0 & \text{on } \Gamma & (b) \\ \dfrac{\partial \phi}{\partial \nu}\Big|_\Gamma = 0 & \text{on } \Gamma & (c) \\ \dfrac{\partial (\Delta \phi)}{\partial \nu}\Big|_\Gamma = 0 & \text{on } \Gamma & (d) \end{cases} \tag{2.33}$$

has only the trivial solution $\phi \equiv 0$.

Notes

1. Since $A^{-3/2}x = A^{-3/2}e_n = r^{-3/2}e_n$; (2.33d) follows from (2.31).

2. The above Lemma is not covered by standard elliptic theory since only three boundary conditions, instead of four, are involved for the fourth-order elliptic operator in (2.33).

3. Recall that as given in Appendix C, if $h(x) \equiv (x - x_0)$ then

$$H(x) = 1 \quad \text{(identity matrix)} \tag{2.34a}$$

$$\operatorname{div} h = n = \dim \Omega \tag{2.34b}$$

4. If ϕ is a solution of (2.33) then it is immediate that via multiplication by $\Delta\phi$ and $\int_\Omega d\Omega$ we have

$$\int_\Omega |\nabla(\Delta\phi)|^2 d\Omega = \lambda \int_\Omega |\nabla\phi|^2 d\Omega \tag{2.35}$$

Proof of Lemma 2.7

Multiply both sides of (2.33a) by $h \cdot \nabla(\Delta\phi)$ and integrate $\int_\Omega d\Omega$.

Left hand side: We have by (A.8), (2.33d), (2.34a-b), (2.35)

$$\int_\Omega \Delta(\Delta\phi) h \cdot \nabla(\Delta\phi) d\Omega = -\frac{1}{2}\int_\Gamma |\nabla(\Delta\phi)|^2 h \cdot \nu d\Gamma + (\frac{n}{2} - 1)\lambda \int_\Omega |\nabla\phi|^2 d\Omega \tag{2.36}$$

Right hand side:

$$\lambda \int_\Omega \phi h \cdot \nabla(\Delta\phi) d\Omega$$

by (A.5)

$$= \lambda \int_\Gamma \Delta\phi \phi h \cdot \nu d\Gamma - \lambda \int_\Omega \Delta\phi \operatorname{div}(\phi h) d\Omega$$

by (A.3),(2.33b),(2.34b)

$$= -\lambda \int_\Omega \Delta\phi h \cdot \nabla\phi d\Omega - n\lambda \int_\Omega \Delta\phi\phi d\Omega$$

by (A.8),(2.34a,b),(A.1)

$$= -\lambda \int_\Gamma \frac{\partial\phi}{\partial\nu} h \cdot \nabla\phi d\Gamma + \frac{\lambda}{2}\int_\Gamma |\nabla\phi| h \cdot \nu d\Gamma + \lambda \int_\Omega |\nabla\phi|^2 d\Omega - \frac{n\lambda}{2}\int_\Omega |\nabla\phi|^2 d\Omega$$
$$- n\lambda \int_\Gamma \frac{\partial\phi}{\partial\nu}\phi d\Gamma + n\lambda \int_\Omega |\nabla\phi|^2 d\Omega$$

by(2.33b-c)

$$= \frac{\lambda}{2}\int_\Gamma |\nabla\phi|^2 h \cdot \nu d\Gamma + (\frac{n}{2} + 1)\lambda \int_\Omega |\nabla\phi|^2 d\Omega$$

But since $\phi\Big|_\Gamma = 0$ we have $|\nabla\phi| = |\frac{\partial\phi}{\partial\nu}| = 0$. Therefore,

$$\lambda \int_\Omega \phi h \cdot \nabla(\Delta\phi) d\Omega = (\frac{n}{2} + 1)\lambda \int_\Omega |\nabla\phi|^2 d\Omega \tag{2.37}$$

Setting (2.36) = (2.37) and simplifying we get

$$-\frac{1}{2}\int_\Gamma |\nabla(\Delta\phi)|^2 h \cdot \nu d\Gamma = 2\lambda \int_\Omega |\nabla\phi|^2 d\Gamma \tag{2.38}$$

Now if we analyze (2.38) we see that by (2.32) the left hand side is nonpositive and the right hand side is nonnegative, therefore both sides must equal zero. This then implies $\nabla\phi \equiv 0$ a.e. and hence, $\phi \equiv c$ a.e. But we have that ϕ eigenfunction implies ϕ smooth, and together with $\phi\Big|_\Gamma \equiv 0$ we have $\phi \equiv 0$. Therefore, Lemma 2.7 and hence Theorem 2.6 are proved. □

Now that we have proven that $\mathbf{A}$ generates a C_0-semigroup of contractions $e^{\mathbf{A}t}$ on $\mathbf{Z}$, it follows that

$$E(t) \le E(0) \quad \text{for} \quad t \ge 0 \tag{2.39}$$

This fact will be used crucially below. The next corollary is a consequence of the dissipative feedback perturbation on the boundary.

Corollary 2.8

By choosing $w\Big|_{\Sigma} = g_1 = -G_1^* A^{-1/2} w_t$, it follows that $G_1^* A^{-1/2} w_t \in L^2((0,\infty); L^2(\Gamma))$ and in fact

$$\| G_1^* A^{-1/2} w_t \|^2_{L^2(\Sigma)} = \int_0^\infty \| G_1^* A^{-1/2} w_t \|^2_\Gamma dt \le \frac{1}{2} E(0) \tag{2.40}$$

for all initial conditions $[w_0, w_1] \in Z$.

Proof of Corollary 2.8

Let $[w_0, w_1] \in \mathcal{D}(\mathbf{A})$ and recall for convenience

$$E(t) = \left\| \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} \right\|^2_Z = \left\| e^{\mathbf{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \right\|^2_Z \quad \text{for } t \ge 0 \tag{2.41}$$

Now $\frac{1}{2}\frac{d}{dt}E(t) = \frac{1}{2}\frac{d}{dt} < e^{\mathbf{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, e^{\mathbf{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} >_Z$

$$= \mathrm{Re} < \mathbf{A} e^{\mathbf{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, e^{\mathbf{A}t} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} >_Z = \mathrm{Re} < \mathbf{A} \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix}, \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix} >_Z$$

via the proof of Lemma 2.5

$$= - \| G_1^* A^{-1/2} w_t \|^2_\Gamma \le 0 \tag{2.42}$$

Remark 2.9

We see that (2.42) shows that such a choice of g_1 does lead to an energy decrease as was demonstrated in another way (using Green's formula) in Remark 2.2. □

Continuing the proof now we integrate $\int_0^\infty dt$ both sides to obtain:

$$\int_0^\infty \| G_1^* A^{-1/2} w_t \|_\Gamma^2 \, dt = -\frac{1}{2} \lim_{T\to\infty} \int_0^T \frac{d}{dt} E(t) dt = \frac{1}{2} \lim_{T\to\infty} \int_T^0 \frac{d}{dt} E(t) dt$$
$$= \frac{1}{2} E(0) - \frac{1}{2} \lim_{T\to\infty} E(T) \le E(0)$$

where in the last inequality we used the contraction of the semigroup, i.e., (2.39). Extension by continuity yields (2.38) for all $[w_0, w_1] \in \mathcal{Z}$. □

Theorem 2.10

Let the radial vector field assumption (2.20) on Ω hold. Then for any $[w_0, w_1] \in \mathcal{Z}$ we have that

$$E(t) = \| \begin{vmatrix} w(t\,,\ w_0\,,\ w_1) \\ w_t(t\,,\ w_0\,,\ w_1) \end{vmatrix} \|_{\mathcal{Z}}^2 = \| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|_{\mathcal{Z}}^2 \to 0 \quad \text{as } t \to +\infty \tag{2.43}$$

Proof of Theorem 2.10

The above result follows by appealing to the Nagy-Foias-Fogel decomposition theory [L]. Since $e^{\mathbf{A}t}$ is a C_0-contraction semigroup by Theorem 2.6, the Hilbert space $\mathcal{Z}$ can be decomposed in a unique way into the orthogonal sum;

$$\mathcal{Z} = \mathcal{Z}_{cnu} \oplus \mathcal{Z}_u \tag{2.44}$$

where both $\mathcal{Z}_{cnu}$ and $\mathcal{Z}_u$ are reducing subspaces for $e^{\mathbf{A}t}$ and its adjoint.

It is also true that

(i) on $\mathcal{Z}_{cnu}$, $e^{\mathbf{A}t}$ is completely nonunitary and weakly stable

(ii) on $\mathcal{Z}_u$, $e^{\mathbf{A}t}$ is a C_0-unitary group.

In our case, $\mathcal{Z}_u = \{0\}$, the trivial subspace, because otherwise an application of Stone's theorem [P] would guarantee at least one eigenvalue of $\mathbf{A}$ on the imaginary axis, but this is clearly false due to Theorem 2.6. Hence $\mathcal{Z} \equiv \mathcal{Z}_{cnu}$ and therefore $e^{\mathbf{A}t}$ is weakly stable on $\mathcal{Z}$. However, since $\mathbf{A}$ has compact resolvent, it follows that $e^{\mathbf{A}t}$ is stable in the strong topology of $\mathcal{Z}[B]$. Therefore, $e^{\mathbf{A}t} z \to 0$ as $t \to +\infty$ for all $z \in \mathcal{Z}$ and strong stability is verified. □

CHAPTER 3
UNIFORM STABILIZATION

3.1 Preliminaries, Change of Variables

Recall our feedback system

$$\begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } Q = (0,\infty)\times\Omega \quad (a) \\ w(0,\cdot) = w_0 \quad w_t(0,\cdot) = w_1 & \text{in } \Omega \quad (b) \\ w\Big|_\Sigma = g_1 = -\dfrac{\partial}{\partial\nu}[\Delta(A^{-3/2}w_t)] & \text{on } \Sigma = (0,\infty)\times\Gamma \quad (c) \\ \dfrac{\partial w}{\partial\nu}\Big|_\Sigma = g_2 = 0 & \text{on } \Sigma \quad (d) \end{cases} \tag{3.0}$$

and the corresponding "energy" $E(t)$ defined by the squared norm of the semigroup

$$E(t) = E(w,t) = \left\| e^{\mathbf{A}t}\begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \right\|_Z^2 = \left\| \begin{vmatrix} w(t) \\ w_t(t) \end{vmatrix} \right\|_Z^2 = \| A^{-1/4} w \|_\Omega^2 + \| A^{-3/4} w_t \|_\Omega^2 \tag{3.1}$$

We want to show that under suitable assumptions on Ω the energy $E(t)$ decays uniformly, exponentially to zero. More precisely there exists constants $C, \delta > 0$ such that for any $[\, w_0, w_1 \,] \in \mathcal{Z} \equiv [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]'$ the corresponding solution of (3.0) satisfies

$$E(t) \le Ce^{-\delta t}E(0) \quad \text{for all } t \ge 0 \tag{3.2}$$

The proof of (3.2) will require a different approach than was used in Chapter 2 to achieve strong stability. The proof will involve multipliers, but before we can proceed a change of variables must be initiated. Let

$$p(t) = A^{-3/2}w_t(t) \quad \text{for } [\, w_0, w_1 \,] \in \mathcal{Z} \tag{3.3}$$

Then since $w_t(t) \in [\mathcal{D}(A^{3/4})]'$ i.e. $A^{-3/4}w_t(t) \in L^2(\Omega)$, it follows that $A^{3/4}p(t) = A^{3/4}A^{-3/2}w_t(t) = A^{-3/4}w_t(t) \in L^2(\Omega)$. Thus

$$p(t) \in \mathcal{D}(A^{3/4}) = \left\{ f \in H^3(\Omega) : f\Big|_\Gamma = \frac{\partial f}{\partial \nu}\Big|_\Gamma = 0 \right\}$$

So in particular

$$\begin{cases} p\Big|_\Sigma = 0 & (a) \\ \dfrac{\partial p}{\partial \nu}\Big|_\Sigma = 0 & (b) \end{cases} \tag{3.4}$$

Now recalling (2.15) we have using (3.3)

$$p_t = A^{-3/2} w_{tt} = A^{-3/2}[-Aw - AG_1 G_1^* A^{-1/2} w_t] \tag{3.5}$$

$$= -A^{-1/2} w - A^{-1/2} G_1 G_1^* A^{-1/2} w_t \tag{3.6}$$

Differentiating once more in time and using $A^{-1/2} w_t = Ap$ and $A^{-1/2} w_{tt} = Ap_t$ we get

$$p_{tt} = -Ap - A^{-1/2} G_1 G^* A p_t \tag{3.6}$$

and hence

$$\begin{cases} p_{tt} + \Delta^2 p = -A^{-1/2} G_1 G^* A p_t & \text{in } Q \quad (a) \\ p(0,\cdot) = p_0 = A^{-3/2} w_1, \quad p_t(0,\cdot) = p_1 = -A^{-1/2} w_0 - A^{-1/2} G_1 G_1^* A^{-1/2} w_1 & \text{in } \Omega \quad (b) \\ p\Big|_\Sigma = 0 & \text{on } \Sigma \quad (c) \\ \dfrac{\partial p}{\partial \nu}\Big|_\Sigma = 0 & \text{on } \Sigma \quad (d) \end{cases} \tag{3.7}$$

3.2 Uniform Stabilization

Our main result referred to in Chapter 1 as Theorem 1.3 is as follows:

Theorem 3.1

Assume that there exists a point $x_0 \in \mathbb{R}^n$ so that the radial vector field defined by $h(x) = (x - x_0)$ satisfies

$$h(x) \cdot \nu \geq \gamma > 0 \quad \text{on } \Gamma \tag{3.8}$$

where ν is the unit outward normal, then there exists positive constants C, δ such that

$$E(t) \leq Ce^{-\delta t} E(0) \quad \text{for all } t \geq 0 \tag{3.9}$$

The proof of Theorem 3.1 will follow directly from the next theorem, by taking $\lim_{\beta \to 0}$ in (3.10) and invoking Datko's theorem [D.1].

Theorem 3.2

Under the same assumption (3.8) as in the above theorem, there exists a constant K such that for all initial data $[w_0, w_1] \in \mathcal{Z}$ (therefore yielding $E(0) < +\infty$) and all $0 < \beta \leq 1$ it follows that

$$\int_0^\infty e^{-2\beta t} E(t)dt = \int_0^\infty e^{-2\beta t} \left\| \begin{matrix} w(t) \\ w_t(t) \end{matrix} \right\|_{\mathcal{Z}}^2 dt = \int_0^\infty e^{-2\beta t} \left\| e^{\mathbf{A}t} \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_{\mathcal{Z}}^2 dt \leq KE(0) \tag{3.10}$$

Proof of Theorem 3.2 (A multiplier approach)

First, we take initial data smooth $[w_0, w_1] \in \mathcal{D}(\mathbf{A})$ (see 2.17b), which then guarantees $[w(t), w_t(t)] \in C([0,T]; \mathcal{D}(\mathbf{A}))$ for all $T < \infty$ and find the desired estimate with K independent of $[w_0, w_1]$ and of β. Then extension by continuity yields (3.10) for all initial data $[w_0, w_1] \in \mathcal{Z}$.

Now we recall our starting identity which we derived in the appendices via the multipliers $e^{-2\beta t} h \cdot \nabla(\Delta p)$ and $e^{-2\beta t} \Delta p \operatorname{div} h$ (recall (C.8)):

$$\begin{aligned}
&\int_\Sigma e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} h \cdot \nabla(\Delta p) d\Sigma - \frac{1}{2} \int_\Sigma e^{-2\beta t} |\nabla(\Delta p)|^2 h \cdot \nu d\Sigma + \frac{n}{2} \int_\Sigma e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} \Delta p d\Sigma \\
&= \int_Q e^{-2\beta t} |\nabla p_t|^2 dQ + \int_Q e^{-2\beta t} |\nabla(\Delta p)|^2 dQ - 2\beta \int_Q e^{-2\beta t} p_t h \cdot \nabla(\Delta p) dQ \\
&- n\beta \int_Q e^{-2\beta t} p_t \Delta p dQ - \int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A^{-1/2} p_t h \cdot \nabla(\Delta p) dQ \\
&- \frac{n}{2} \int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A p_t \Delta p dQ - \lim_{T \to \infty} \left[e^{-2\beta t} < p_t, \; h \cdot \nabla(\Delta p) >_\Omega \right]_0^T \\
&- \frac{n}{2} \lim_{T \to \infty} \left[e^{-2\beta t} < p_t, \; \Delta p >_\Omega \right]_0^T \qquad (3.11a) \\
&= \int_Q e^{-2\beta t} |\nabla p_t|^2 dQ + \int_Q e^{-2\beta t} |\nabla(\Delta p)|^2 dQ - Q_1 - Q_2 - Q_3 - Q_4 - L_1 - L_2 \qquad (3.11b)
\end{aligned}$$

where the Q_i, $i = 1,2,3,4$ are the corresponding integrals over Q and the L_i, $i = 1,2$ are the corresponding limit terms. The first two integrals are most important as we shall see below. Next we prove:

Lemma 3.3

There exists a constant C_1 such that

$$\| \Delta p \|_\Gamma^2 \leq C_1 \| A^{3/4} p \|_\Omega^2 \tag{3.12}$$

Proof

First by trace theory [L-M] $\| \Delta p \|_{\Gamma}^2 \leq C \| \Delta p \|_{H^{1/2}(\Omega)}^2$

$$\leq C \| \Delta p \|_{H^1(\Omega)}^2 \quad \text{since } H^1(\Omega) \text{ is a stronger topology}$$

$$= C \| \Delta p \|_{\Omega}^2 + C \| \left| \nabla(\Delta p) \right| \|_{\Omega}^2 \quad \text{by definition}$$

$$\leq C \| A^{1/2} p \|_{\Omega}^2 + CCp_3 \| A^{3/4} p \|_{\Omega}^2 \quad \text{by norm equivalaence}$$

$$\leq C \| A^{-1/4} p \|_{L(\Omega)}^2 \| A^{3/4} p \|_{\Omega}^2 + CCp_3 \| A^{3/4} p \|_{\Omega}^2 = C_1 \| A^{3/4} p \|_{\Omega}^2 \quad \square$$

We continue the proof of Theorem 3.2 which will be divide into seven steps.

Step 1 : Absorption of the boundary terms in (3.11). Let $M_{hb} = \max_{\Gamma} |h|^2$ and choose $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that

$$\varepsilon_1 M_{hb} < \frac{\gamma}{2} \tag{3.13a}$$

$$\varepsilon_2 \frac{nC_1}{2} < \alpha_1 \tag{3.13b}$$

where C_1 as in Lemma 3.3 and α_1 to be given below.

Before proceeding, note that the following inequality will be used extensively below:

$$\text{For any } \varepsilon > 0 \; ; \; 2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \tag{*}$$

Next we operate on the left hand side (L.H.S) of equation (3.11) by using (*) with ε_1 and ε_2 as well as the assumption on $h \cdot \nu$ to obtain:

$$\int_{\Sigma} e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} h \cdot \nabla(\Delta p) d\Sigma - \frac{1}{2} \int_{\Sigma} e^{-2\beta t} |\nabla(\Delta p)|^2 h \cdot \nu d\Sigma + \frac{n}{2} \int_{\Sigma} e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} \Delta p d\Sigma$$

$$\leq \frac{1}{\varepsilon_1} \int_{\Sigma} e^{-2\beta t} |\frac{\partial(\Delta p)}{\partial \nu}|^2 d\Sigma + \varepsilon_1 M_{hb} \int_{\Sigma} e^{-2\beta t} |\nabla(\Delta p)|^2 d\Sigma - \frac{\gamma}{2} \int_{\Sigma} e^{-2\beta t} |\nabla(\Delta p)|^2 d\Sigma$$

$$+ \frac{n}{2\varepsilon_2} \int_{\Sigma} e^{-2\beta t} |\frac{\partial(\Delta p)}{\partial \nu}|^2 d\Sigma + \varepsilon_2 \frac{n}{2} \int_{\Sigma} e^{-2\beta t} \| \Delta p \|_{\Gamma}^2 d\Sigma$$

$$\leq (\frac{1}{\varepsilon_1} + \varepsilon_2 \frac{n}{2}) E(0) + (\varepsilon_1 M_{hb} - \frac{\gamma}{2}) \int_{\Sigma} e^{-2\beta t} |\nabla(\Delta p)|^2 d\Sigma + \frac{\varepsilon_2 n C_1}{2} \int_0^{\infty} e^{-2\beta t} \| A^{3/4} p \|_{\Omega}^2 dt$$

where the E(0) term follows by using (2.40) and we have also used Lemma 3.3 for the Δp term.

Now we can drop the middle term because by (3.13a) $\varepsilon_1 M_{hb} - \frac{\gamma}{2} < 0$ and use the fact that

$\| A^{3/4} p \|_{\Omega}^2 = \| A^{-3/4} w_t \|_{\Omega}^2 \leq E(t)$ to obtain:

$$\text{(L.H.S.) of (3.11)} \quad \leq K_1 E(0) + \frac{\varepsilon_2 n C_1}{2} \int_0^\infty e^{-2\beta t} E(t)dt \tag{3.14}$$

where $K_1 = \frac{1}{\varepsilon_1} + \frac{n}{2\varepsilon_2}$.

Finally, utilizing (3.14) in (3.11) and isolating the important terms we obtain:

$$\int_Q e^{-2\beta t} |\nabla p_t|^2 dQ + \int_Q e^{-2\beta t} |\nabla(\Delta p)|^2 dQ \leq K_1 E(0) + \frac{\varepsilon_2 n C_1}{2} \int_0^\infty e^{-2\beta t} E(t)dt + Q_1 + Q_2 + Q_3 + Q_4 + L_1 + L_2 \tag{3.15}$$

Step 2 : Isolating the energy integral $\int_0^\infty e^{-2\beta t} E(t)dt$. First since $p_t \in \mathcal{D}(A^{1/4})$ there exists a constant $Cp_1 > 0$ such that

$$|| \left|\nabla p_t\right| ||_\Omega^2 \geq Cp_1 || A^{1/4} p_t ||_\Omega^2 \text{[L-T.2]} \tag{3.16}$$

Similarly $p \in \mathcal{D}(A^{3/4})$ implies the existence of $Cp_3 > 0$ such that

$$|| \left|\nabla(\Delta p)\right| ||_\Omega^2 \geq Cp_3 || A^{3/4} p ||_\Omega^2 \text{[L-T.2]} \tag{3.17}$$

Next using (3.15) and (3.16) we have

$$|| \left|\nabla p_t\right| ||_\Omega^2 \geq Cp_1 || A^{1/4} p_t ||_\Omega^2$$

$$= Cp_1 || A^{-1/4} w ||_\Omega^2 + 2Cp_1 \text{Re} < A^{-1/4} w \,,\, A^{-1/4} G_1 G_1^* A^{-1/2} w_t >_\Omega + Cp_1 || A^{-1/4} G_1 G_1^* A^{-1/2} w_t ||_\Omega^2$$

Now to bound below we use $2ab \geq -\varepsilon a^2 - \frac{1}{\varepsilon} b^2$ on the middle term and taking $\varepsilon = \frac{1}{2}$ we obtain

$$|| \left|\nabla p_t\right| ||_\Omega^2 \geq \frac{Cp_1}{2} || A^{-1/4} w ||_\Omega^2 - Cp_1 || A^{-1/4} G_1 G_1^* A^{-1/2} w_t ||_\Omega^2 \tag{3.18}$$

Also it is immediate that

$$|| \left|\nabla(\Delta p)\right| ||_\Omega^2 \geq Cp_3 || A^{3/4} p ||_\Omega^2 = Cp_3 || A^{-3/4} w_t ||_\Omega^2 \tag{3.19}$$

Putting the pieces together in a Lemma:

Lemma 3.4

$$\text{Let} \quad \alpha_1 = \min \left\{ \frac{Cp_1}{2}, Cp_3 \right\} \tag{3.20}$$

$$\text{and} \quad K_2 = Cp_1 || A^{-1/2} G_1 ||^2_{\mathcal{L}(\Gamma, \Omega)} \tag{3.21}$$

$$\text{Then} \quad \int_Q e^{-2\beta t} |\nabla p_t|^2 dQ + \int_Q e^{-2\beta t} |\nabla(\Delta p)|^2 dQ \geq \alpha_1 \int_0^\infty e^{-2\beta t} E(t)dt - K_2 E(0) \tag{3.22}$$

Proof

$$\int_Q e^{-2\beta t} |\nabla(\Delta p)|^2 dQ + \int_Q e^{-2\beta t} |\nabla p_t|^2 dQ$$

by (3.18), (3.19)

$$\geq Cp_3 \int_0^\infty e^{-2\beta t} \| A^{-3/4} w_t \|_\Omega^2 dt + \frac{Cp_1}{2} \int_0^\infty e^{-2\beta t} \| A^{-1/4} w \|_\Omega^2 dt - Cp_1 \int_0^\infty e^{-2\beta t} \| A^{-1/4} G_1 G_1^* A^{-1/2} w_t \|_\Omega^2 dt$$

$$\geq \alpha_1 \int_0^\infty e^{-2\beta t} \left[\| A^{-1/4} w \|_\Omega^2 + \| A^{-3/4} w_t \|_\Omega^2 \right] dt - Cp_1 \| A^{-1/4} G_1 \|^2_{\mathcal{L}(\Gamma, \Omega)} \int_0^\infty \| G_1^* A^{-1/2} w_t \|_\Gamma^2 dt$$

$$\geq \text{by (2.40)} \quad \alpha_1 \int_0^\infty e^{-2\beta t} E(t) dt - Cp_1 \| A^{-1/4} G_1 \|^2_{\mathcal{L}(\Gamma, \Omega)} E(0). \quad \square$$

Collecting our results so far we have via (3.13b), (3.15), (3.22) that

$$\alpha_2 \int_0^\infty e^{-2\beta t} E(t) dt \leq K_3 E(0) + L_1 + L_2 + Q_1 + Q_2 + Q_3 + Q_4 \tag{3.23}$$

where

$$K_3 = K_1 + K_2 \tag{3.24a}$$

and

$$\alpha_2 = \alpha_1 - \varepsilon_2 \frac{nC_1}{2} > 0 \tag{3.24b}$$

Step 3 : Handling the limit terms arising from integration by parts in time.

Remark 3.5

The following will be used in evaluating the limit terms in (3.23). Since we are taking initial conditions $[w_0, w_1] \in \mathcal{D}(\mathbf{A})$ we have by the regularity theory [L-T.5], $[w(t), w_t(t)] \in C([0,T]; \mathcal{D}(\mathbf{A}))$, i.e., $w(t) \in C([0,T]; H^1(\Omega))$ and $w_t(t) \in C([0,T]; [\mathcal{D}(A^{1/4})]')$.

Computing $\left\| \begin{matrix} w \\ w_t \end{matrix} \right\|_{\mathcal{D}(\mathbf{A})} = \left\| e^{\mathbf{A}t} \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_{\mathcal{D}(\mathbf{A})}$

$$= \left\| \mathbf{A} e^{\mathbf{A}t} \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_Z = \left\| e^{\mathbf{A}t} \mathbf{A} \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_Z \leq \left\| \mathbf{A} \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|_Z \tag{3.25}$$

where we have used $0 \in \rho(\mathbf{A})$ quoted in Theorem 2.6. Hence we have that

$$\| w \|^2_{H^1(\Omega)} + \| A^{-1/4} w_t \|_\Omega^2 \leq \left\| \mathbf{A} \begin{matrix} w_0 \\ w_1 \end{matrix} \right\|^2_Z \quad \text{for all } t \geq 0 \tag{3.26}$$

Now letting $M_h = \max_{\overline{\Omega}} |h|^2$ we have

$$\left| \lim_{T\to\infty} e^{-2\beta T} < p_t(T),\ h\cdot\nabla(\Delta p(T)) >_\Omega \right| \le \lim_{T\to\infty} e^{-2\beta T}\left[\| p_t(T)\|_\Omega^2 + M_h \| A^{3/4} p(T)\|_\Omega^2 \right]$$

by (3.5), (3.9)

$$\le \lim_{T\to\infty} e^{-2\beta T}\Big[2\,\| A^{-1/4}\|^2_{L(\Omega)}\ \| A^{-1/4} w(T)\|_\Omega^2$$

$$+ 2\,\| A^{-1/2} G_1 G_1^* A^{-1/4}\|^2_{L(\Omega)}\ \| A^{-1/4} w_t(T)\|_\Omega^2 + M_h E(0)\Big]$$

by (3.26)

$$\le \lim_{T\to\infty} e^{-2\beta T}\left\{ CE(0) + 2\,\| A^{-1/2} G_1 G_1^* A^{-1/4}\|^2_{L(\Omega)}\ \| \mathbf{A}\Big|_{w_1}^{w_0}\Big\|_{\mathbf{Z}}^2 \right\} = 0 \tag{3.27}$$

A similar computation shows that $\left| \lim_{T\to\infty} e^{-2\beta T} < p_t(T),\ \Delta p(T) >_\Omega \right| = 0$ (3.28)

Hence we have that $L_1 + L_2 = - < p_1,\ h\cdot\nabla(\Delta p_0) >_\Omega - \frac{n}{2} < p_1,\ \Delta p_0 >_\Omega$

$$= < A^{-1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1,\ h\cdot\nabla(\Delta p_0) >_\Omega + \frac{n}{2} < A^{-1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1,\ \Delta p_0 >_\Omega \tag{3.29}$$

Step 4 : Handling the terms premultiplied by β, i.e. the terms Q_1 and Q_2. First

$$\left| Q_1 \right| = \left| 2\beta \int_Q e^{-2\beta t} p_t h\cdot\nabla(\Delta p) dQ \right|$$

$$\le \beta \int_0^\infty e^{-2\beta t}\ \| p_t\|_\Omega^2 + \beta M_h \int_0^\infty e^{-2\beta t}\ \| \left|\nabla(\Delta p)\right| \|_\Omega^2 dt$$

by (3.5)

$$\le 2\beta\ \| A^{-1/4}\|^2_{L(\Omega)} \int_0^\infty e^{-2\beta t}\ \| A^{-1/4} w\|_\Omega^2 dt$$

$$+ 2\beta\ \| A^{-1/2} G_1\|^2_{L(\Gamma,\Omega)} \int_0^\infty e^{-2\beta t}\ \| G_1^* A^{-1/2} w_t\|_\Gamma^2 dt + \beta M_h C p_3 \int_0^\infty e^{-2\beta t}\ \| A^{-3/4} w_t\|_\Omega^2 dt$$

$$\le 2\beta(\ \| A^{-1/4}\|^2_{L(\Omega)} + M_h C p_3) \int_0^\infty e^{-2\beta t} E(t) dt +\ \| A^{-1/2} G_1\|^2_{L(\Gamma,\Omega)} E(0)$$

(where we have used the facts that $\| A^{-1/4} w\|_\Omega^2 \le E(t)$ and similarly

$\| A^{-3/4} w_t \|_{\Omega}^2 \leq E(t)$ as well as (2.40))

$$\leq \left[\| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} + M_h C p_3 + \| A^{-1/2} G_1 \|^2_{\mathbf{L}(\Gamma, \Omega)} \right] E(0) = K_4 E(0) \tag{3.30}$$

where we also have used $E(t) \leq E(0)$ for all $t \geq 0$ (contraction of the semigroup) and

$$\int_0^{\infty} e^{-2\beta t} dt = \frac{1}{2\beta} \tag{3.31}$$

A similar computation shows that there exists a constant K_5 (independent of β) such that

$$\left| Q_2 \right| \leq K_5 E(0) \tag{3.32}$$

Thus we arrive at

Lemma 3.7

$$\text{Let } K_6 = K_3 + K_4 + K_5 \tag{3.33}$$

Then $\alpha_2 \int_0^{\infty} e^{-2\beta t} E(t) dt \leq K_6 E(0) + Q_3 + Q_4 + < A^{1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1 \, , \, h \cdot \nabla(\Delta p_0) >_{\Omega}$

$$+ \frac{n}{2} < A^{-1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1 \, , \, \Delta p_0 >_{\Omega} \tag{3.34}$$

Proof

Follows immediately by using (3.29), (3.30), (3.32) in (3.23). □

Now having dispersed with the "lower order" terms, our task is to absorb the more difficult interior integrals.

Step 5 : The term Q_3. We apply integration by parts in time with $dv = A^{-1/2} G_1 G_1^* A^{-1/2} w_t$, $u = e^{-2\beta t} h \cdot \nabla(\Delta p)$ to obtain

$$\int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A^{-1/2} w_{tt} h \cdot \nabla(\Delta p) dQ = \lim_{T \to \infty} \left[e^{-2\beta t} < A^{-1/2} G_1 G_1^* A^{-1/2} w_t \, , \, h \cdot \nabla(\Delta p) >_{\Omega} \right]_0^T$$

$$+ 2\beta \int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A^{-1/2} w_t h \cdot \nabla(\Delta p) dQ - \int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A^{-1/2} w_t h \cdot \nabla(\Delta p_t) dQ \tag{3.35}$$

Part A

Since $\| A^{-1/2}G_1G_1^*A^{-1/2}w_t(T)\|_\Omega^2 \le \| A^{-1/2}G_1G_1^*A^{-1/4}\|^2_{\mathbf{L}(\Omega)} \| \|A^{-1/4}w_t(T)\|_\Omega^2$

$$\text{by (3.26)} \quad \le \| A^{-1/2}G_1G_1^*A^{-1/4}\|^2_{\mathbf{L}(\Omega)} \| \mathbf{A}\Big|_{w_1}^{w_0}\|_Z^2$$

an argument exactly like the one used to obtain (3.27) yields:

$$\lim_{T\to\infty} e^{-2\beta T} < A^{-1/2}G_1G_1^*A^{-1/2}w_t(T),\ h\cdot\nabla(\Delta p(T))>_\Omega = 0 \tag{3.36}$$

Thus we only have a contribution from t = 0, i.e.,

$$\lim_{T\to\infty}\Big[e^{-2\beta t} < A^{-1/2}G_1G_1^*A^{-1/2}w_t,\ h\cdot\nabla(\Delta p)>_\Omega\Big]_0^T = -< A^{-1/2}G_1G_1^*A^{-1/2}w_1,\ h\cdot\nabla(\Delta p_0)>_\Omega \tag{3.37}$$

Part B

$$\begin{aligned} &2\beta\int_0^\infty e^{-2\beta t} < A^{-1/2}G_1G_1^*A^{-1/2}w_t,\ h\cdot\nabla(\Delta p)>_\Omega dt \\ &\le 2\beta\int_0^\infty e^{-2\beta t}\| A^{-1/2}G_1G_1^*A^{-1/2}w_t\|_\Omega^2 dt + 2\beta M_h\int_0^\infty e^{-2\beta t}\|\nabla(\Delta p)\|_\Omega^2 dt \\ &\le (\,2\| A^{-1/2}G_1\|^2_{\mathbf{L}(\Gamma,\,\Omega)} + M_hCp_3\,)\,E(0) \end{aligned} \tag{3.38}$$

where we have used (2.39), (2.40) and (3.31)

Part C The following Lemma will enable us to handle the remaining term and hence complete Step 5.

Lemma 3.8

For $h \in [C^2(\bar\Omega)]^n$ we have that there exists a constant C_h such that

$$\| A^{-1/2}(h\cdot\nabla(\Delta p_t))\|_\Omega^2 \le C_h\| A^{1/4}p_t\|_\Omega^2 \tag{3.39a}$$

Remark

Since $\mathcal{D}(A^{1/2}) = H_0^2(\Omega)$ (see [L-T.2, Appendix C]) so that $[\mathcal{D}(A^{1/2})]' \simeq H^{-2}(\Omega)$ (with equivalent norms) we have

$$\| A^{-1/2}(h\cdot\nabla(\Delta p_t))\|_\Omega = \| h\cdot\nabla(\Delta p_t)\|_{[\mathcal{D}(A^{1/2})]'} \simeq \| h\cdot\nabla(\Delta p_t)\|_{H^{-2}(\Omega)} \tag{3.39b}$$

Claim Let $z \in H^{-2}(\Omega)$, $h \in C^2(\bar\Omega)$. Then $hz \in H^{-2}(\Omega)$.

Proof of claim (See also [L-M, Vol I, Theorem 7.3, p31])

By assumption $< z,\ g >_\Omega = \int_\Omega zg\,d\Omega < \infty$; $g \in H_0^2(\Omega)$ and we want to show that

$< hz\ ,\ g > = \int_\Omega zhg d\Omega < \infty\ ;\ \ g \in H_0^2(\Omega).$

Thus it suffices to show $hg \in H_0^2(\Omega)$, if $g \in H_0^2(\Omega)$. Clearly $hg \in H^2(\Omega)$. So we must show

$$hg\Big|_\Gamma = \frac{\partial(hg)}{\partial \nu}\Big|_\Gamma = 0$$

but $hg\Big|_\Gamma = h\Big|_\Gamma g\Big|_\Gamma = 0$ since $g \in H_0^2(\Omega)$.

Also $\dfrac{\partial(hg)}{\partial \nu}\Big|_\Gamma = \nabla(hg)\cdot \nu = \left\{ [(hg)_{x_1}\ ,\ (hg)_{x_2}\ ,\ \cdots, (hg)_{x_n}] \cdot [\nu_1, \nu_2, \cdots, \nu_n] \right\}$

$$= \left\{ [h_{x_1} g + hg_{x_1}]\nu_1 + [h_{x_2} g + hg_{x_2}]\nu_2 + \cdots + [h_{x_n} g + hg_{x_n}]\nu_n \right\}\Big|_\Gamma = 0$$

since $g \in H_0^2(\Omega)$ implies $g\Big|_\Gamma = g_{x_1}\Big|_\Gamma = g_{x_2}\Big|_\Gamma = \cdots = g_{x_n}\Big|_\Gamma = 0$ □

Proof of Lemma 3.8

Using the above claim in (3.39b) we have

$$\begin{aligned} \| A^{-1/2}(h\cdot\nabla(\Delta p_t)) \|_\Omega &\approx \| h\cdot\nabla(\Delta p_t)) \|_{H^{-2}(\Omega)} \\ &\le M_h \| \nabla(\Delta p_t) \|_{H^{-2}(\Omega)} \\ &\le C M_h \| p_t \|_{H^1(\Omega)} \quad \text{by [L-M, Vol I]} \\ &\le C M_h C p_1 \| \left|\nabla p_t\right| \|_\Omega \quad \text{by Poincare Inequality} \\ &\le C_h \| A^{1/4} p_t \|_\Omega \quad \text{via norm equivalence} \ \square \end{aligned}$$

Continuing Part C, choose $\varepsilon_3 > 0$ such that $2C_h\varepsilon_3 < \alpha_2$. (3.40)

Now we compute $\left| \int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A^{-1/2} w_t h\cdot\nabla(\Delta p_t) dQ \right|$

$$\le \int_0^\infty e^{-2\beta t} \left| < G_1 G_1^* A^{-1/2} w_t\ ,\ A^{-1/2}(h\cdot\nabla(\Delta p_t)) >_\Omega \right| dt$$

by (3.39a)

$$\le \| G_1 \|^2_{\mathbf{L}(\Gamma,\,\Omega)} \int_0^\infty \| G_1^* A^{-1/2} w_t \|^2_\Gamma dt + C_h \int_0^\infty e^{-2\beta t} \| A^{1/4} p_t \|^2_\Omega dt$$

by (2.40),(*)

$$\leq \| G_1 \|^2_{\mathbf{L}(\Gamma,\,\Omega)} E(0) + \frac{2C_h}{\varepsilon_3}\int_0^\infty e^{-2\beta t} \| A^{-1/4} G_1 G_1^* A^{-1/2} w_t \|^2_\Omega dt + 2C_h\varepsilon_3 \int_0^\infty e^{-2\beta t} \| A^{-1/4} w \|^2_\Omega dt$$

$$\leq \left[\| G_1 \|^2_{\mathbf{L}(\Gamma,\,\Omega)} + \frac{2C_h}{\varepsilon_3} \| A^{-1/4} G_1 \|^2_{\mathbf{L}(\Gamma,\,\Omega)} \right] E(0) + 2C_h\varepsilon_3 \int_0^\infty e^{-2\beta t} E(t) dt$$

$$= K_8 E(0) + 2C_h\varepsilon_3 \int_0^\infty e^{-2\beta t} E(t) dt \tag{3.41}$$

Again organizing our results we have

Lemma 3.9

$$\text{Let } K_9 = K_6 + K_7 + K_8 \tag{3.42}$$

$$\text{and } \alpha_3 = \alpha_2 - 2C_h\varepsilon_3 > 0 \tag{3.43}$$

$$\text{then } \alpha_3 \int_0^\infty e^{-2\beta t} E(t) dt \leq K_9 E(0) + Q_4 + < A^{-1/2} w_0 \,,\; h \cdot \nabla(\Delta p_0) >_\Omega + \frac{n}{2} < A^{-1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1 \,,\; \Delta p_0 >_\Omega \tag{3.44}$$

Proof

The proof follows directly by utilizing (3.36), (3.38) and (3.41) in (3.34) and taking advantage of the cancellation of terms

$$< A^{-1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1 \,,\; h \cdot \nabla(\Delta p_0) >_\Omega - < A^{-1/2} G_1 G_1^* A^{-1/2} w_1 \,,\; h \cdot \nabla(\Delta p_0) >_\Omega$$
$$= < A^{-1/2} w_0 \,,\; h \cdot \nabla(\Delta p_0) >_\Omega \quad \square$$

Before beginning Step 6, the following Lemma will be needed.

Lemma 3.10

The L^2-norm of Δp is bounded by the initial energy $E(0)$, i.e.,

$$\| \Delta p \|^2_\Omega = \| A^{1/2} p \|^2_\Omega = \| A^{-1/4} A^{3/4} p \|^2_\Omega \leq \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} \| A^{3/4} p \|^2_\Omega$$
$$= \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} \| A^{-3/4} w_t \|^2_\Omega \leq \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} E(t) \leq \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} E(0) \tag{3.45}$$

Proof

$$\| A^{1/2}p \|_{\Omega}^2 = < Ap\ ,\ p >_{\Omega} = < \Delta(\Delta p)\ ,\ p >_{\Omega}$$

$$\text{by (A.0)} \quad = < \Delta p\ ,\ \Delta p >_{\Omega} + \int_{\Gamma} \frac{\partial(\Delta p)}{\partial \nu} p d\Gamma - \int_{\Gamma} \Delta p \frac{\partial p}{\partial \nu} d\Gamma$$

$$\text{by (3.4a,b)} = < \Delta p\ ,\ \Delta p >_{\Omega} = \| \Delta p \|_{\Omega}^2 \quad \square$$

Step 6 : The term Q_4. As in Step 5, we begin with integration by parts in time with $dv = A^{-1/2}G_1G_1^*A^{-1/2}w_{tt}$, $u = e^{-2\beta t}\Delta p$ to obtain:

$$\frac{n}{2}\int_Q e^{-2\beta t}A^{-1/2}G_1G_1^*A^{-1/2}w_{tt}\Delta p dQ = \frac{n}{2}\lim_{T\to\infty}\left[e^{-2\beta t} < A^{-1/2}G_1G_1^*A^{-1/2}w_t\ ,\ \Delta p >_{\Omega}\right]_0^T$$
$$+ n\beta\int_Q e^{-2\beta t}A^{-1/2}G_1G_1^*A^{-1/2}w_t\Delta p dQ - \frac{n}{2}\int_Q e^{-2\beta t}A^{-1/2}G_1G_1^*A^{-1/2}w_t\Delta p_t dQ \qquad (3.46)$$

Now we begin a three part process as in step 5.

Part A Using (3.26) and (3.45) it follows directly that

$$\frac{n}{2}\lim_{T\to\infty} e^{-2\beta T} < A^{-1/2}G_1G_1^*A^{-1/2}w_t(T)\ ,\ \Delta p(T) >_{\Omega} \le \frac{n}{2}C\lim_{T\to\infty} e^{-2\beta T} = 0 \qquad (3.47)$$

So again we have only a contribution from $t = 0$ i.e.

$$\frac{n}{2}\lim_{T\to\infty}\left[e^{-2\beta t} < A^{-1/2}G_1G_1^*A^{-1/2}w_t\ ,\ \Delta p >_{\Omega}\right]_0^T = -\frac{n}{2} < A^{-1/2}G_1G_1^*A^{-1/2}w_1\ ,\ \Delta p_0 >_{\Omega} \qquad (3.48)$$

Part B

$$n\beta\int_0^{\infty} e^{-2\beta t} < A^{-1/2}G_1G_1^*A^{-1/2}w_t\ ,\ \Delta p >_{\Omega} dt$$
$$\le n\beta \| A^{-1/2}G_1 \|_{\mathbf{L}(\Gamma,\,\Omega)}^2 \int_0^{\infty} e^{-2\beta t} \| G_1A^{-1/2}w_t \|_{\Gamma}^2 dt + n\beta\int_0^{\infty} e^{-2\beta t} \| \Delta p \|_{\Omega}^2 dt$$
$$\text{by (2.40), (3.32), (3.45)} \le \left[n \| A^{-1/2}G_1 \|_{\mathbf{L}(\Gamma,\,\Omega)}^2 + \frac{n}{2} \| A^{-1/4} \|_{\mathbf{L}(\Omega)}^2 \right] E(0) = K_{10}E(0) \qquad (3.49)$$

Part C For C_2 as given below, choose $\varepsilon_4 > 0$ such that

$$\varepsilon_4 n C_2 \| A^{-1/4} \|_{\mathbf{L}(\Omega)}^2 < \alpha_3 \qquad (3.50)$$

$$\left| -\frac{n}{2}\int_Q e^{-2\beta t} < A^{-1/2}G_1G_1^*A^{-1/2}w_t\ ,\ \Delta p_t >_{\Omega} dQ \right|$$

$$\le \frac{n}{2}\int_Q e^{-2\beta t}\left| < G_1G_1^*A^{-1/2}w_t\ ,\ A^{-1/2}(\Delta p_t) >_{\Omega} \right| dQ$$

$$\le \frac{n}{2\varepsilon_4}\int_0^{\infty} e^{-2\beta t} \| G_1G_1^*A^{-1/2}w_t \|_{\Omega}^2 dQ + \frac{\varepsilon_4 n}{2}\int_0^{\infty} e^{-2\beta t} \| A^{-1/2}(\Delta p_t) \|_{\Omega}^2 dQ \qquad (3.51)$$

Examine the integral on the right in (3.51)

Using the facts that $\mathcal{D}(A^{1/2}) = H_0^2(\Omega)$ [L-T.2 , Appendix C] (3.52a)

$$\text{and } \| \Delta f \|^2_{H^{-2}(\Omega)} \leq C_2 \| f \|^2_{\Omega} \quad \textit{[L-M, Vol. 1]} \tag{3.52b}$$

We have $\| A^{-1/2} \Delta p_t \|^2_{\Omega} = \| \Delta p_t \|^2_{[\mathcal{D}(A^{1/2})]'} = \| \Delta p_t \|^2_{[H_0^2(\Omega)]'}$

$$= \| \Delta p_t \|^2_{H^{-2}(\Omega)} \leq C_2 \| p_t \|^2_{\Omega} \leq C_2 \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} \| A^{1/4} p_t \|^2_{\Omega} \tag{3.53}$$

so continuing (3.51) $\leq \frac{n}{2\varepsilon_4} \| G_1 \|^2_{\mathbf{L}(\Gamma, \Omega)} E(0) + \frac{\varepsilon_4 n C_2}{2} \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} \int_0^{\infty} e^{-2\beta t} \| A^{1/4} p_t \|^2_{\Omega} dt$

$$\leq \frac{n}{2\varepsilon_4} \| G_1 \|^2_{\mathbf{L}(\Gamma, \Omega)} E(0) + \varepsilon_4 n C_2 \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} \int_0^{\infty} e^{-2\beta t} E(t) dt$$

$$+ \varepsilon_4 n C_2 \| A^{-1/4} G_1 \|^2_{\mathbf{L}(\Gamma, \Omega)} E(0)$$

$$= K_{11} E(0) + \varepsilon_4 n C_2 \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} \int_0^{\infty} e^{-2\beta t} E(t) dt \tag{3.54}$$

Lemma 3.11

Let $K_{12} = K_9 + K_{10} + K_{11}$ (3.55)

and $\alpha_4 = \alpha_3 - \varepsilon_4 n C_2 \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} > 0$ (3.56)

Then $\alpha_4 \int_0^{\infty} e^{-2\beta t} E(t) dt \leq K_{12} E(0) + < A^{-1/2} w_0 , h \cdot \nabla(\Delta p_0) >_{\Omega}$

$$+ \frac{n}{2} < A^{-1/2} w_0 , \Delta p_0 >_{\Omega} \tag{3.57}$$

Proof

The proof follows again directly using (3.48), (3.49), (3.54) and again utilizing the cancellation of terms:

$$\frac{n}{2} < A^{-1/2} w_0 + A^{-1/2} G_1 G_1^* A^{-1/2} w_1 , \Delta p_0 >_{\Omega} - \frac{n}{2} < A^{-1/2} G_1 G_1^* A^{-1/2} w_1 , \Delta p_0 >_{\Omega}$$

$$= \frac{n}{2} < A^{-1/2} w_0 , \Delta p_0 >_{\Omega} \quad \square$$

Step 7 : The initial conditions.

First, $| < A^{-1/2} w_0 , h \cdot \nabla(\Delta p_0) >_{\Omega} | \leq \| A^{-1/2} w_0 \|^2_{\Omega} + M_h C p_3 \| A^{3/4} w_1 \|^2_{\Omega}$

$$\leq \left[\| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} + M_h C p_3 \right] E(0) = K_{13} E(0) \tag{3.58}$$

Second, $|\frac{n}{2} < A^{-1/2}w_0\,,\ \Delta p_0 >_\Omega| \le \frac{n}{2}\left[\ \| A^{-1/2}w_0 \|^2_\Omega + \| \Delta p_0 \|^2_\Omega\right]$

$$\le \frac{n}{2}\left[\ \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} E(0) + \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} E(0)\right] = n \| A^{-1/4} \|^2_{\mathbf{L}(\Omega)} E(0) = K_{14}E(0) \qquad (3.59)$$

where we have applied Lemma 3.10 for the Δp_0 term.
Finally using (3.58), (3.59) in (3.57) we have

$$\alpha_4 \int_0^\infty e^{-2\beta t} E(t)dt \le K_{15}E(0) \qquad (3.60)$$

$$\text{where } K_{15} = K_{12} + K_{13} + K_{14} \qquad (3.61)$$

Dividing (3.60) by $\alpha_4 > 0$ and letting $K = \frac{K_{15}}{\alpha_4}$ Theorem 3.2 is proved. □

Now the proof of the main result (Theorem 3.1) will follow easily from the below quoted result.

Theorem (Datko) [B, page 176]

Suppose $T(t)$ is a semigroup on a Banach space $\boldsymbol{X}$. If for all $x \in \boldsymbol{X}$

$$\int_0^\infty \| T(t)x \|^2_{\boldsymbol{X}} dt < \infty$$

then the semigroup is uniformly (exponentially) stable, i.e., there exists constants $C, \delta \ge 0$ such that

$$\| T(t) \|_{\mathbf{L}(X)} \le Ce^{-\delta t} \quad \text{for all } t \ge 0 \quad \square$$

Proof of Theorem 3.1

We have seen that $\int_0^\infty e^{-2\beta t} E(t)dt \le K\,E(0)$ where K is a constant independent of $\beta \in (0,1]$ and of $[\,w_0, w_1\,] \in \mathcal{D}(\mathbf{A})$. Thus taking the $\lim_{\beta \to 0}$ we get

$$\int_0^\infty E(t)\,dt \le K\,E(0) \qquad (3.62)$$

But $E(t)$ is simply the squared norm of the semigroup applied to any initial conditions $[\,w_0, w_1\,] \in \mathcal{D}(\mathbf{A})$ and hence by continuity all $[\,w_0, w_1\,] \in \boldsymbol{Z}$, i.e,

substituting $E(t) = \| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|^2_{\boldsymbol{Z}}$ into (3.62),

we get $\int_0^\infty \| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|^2_{\boldsymbol{Z}} dt \le K\,E(0) < \infty$

Thus by Datko's Theorem there exists $C_1, \delta_1 > 0$ such that

$$\| e^{\mathbf{A}t} \|_{\mathbf{L}(\mathbf{Z})} \leq C_1 e^{-\delta_1 t} \quad \text{and hence} \quad \| e^{\mathbf{A}t} \|^2_{\mathbf{L}(\mathbf{Z})} \leq C_1^2 e^{-2\delta_1 t}$$

therefore by taking $C = C_1^2$ and $\delta = 2\delta_1$ we have

$$E(t) = \| e^{\mathbf{A}t} \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|^2_{\mathbf{Z}} \leq \| e^{\mathbf{A}t} \|^2_{\mathbf{L}(\mathbf{Z})} \| \begin{vmatrix} w_0 \\ w_1 \end{vmatrix} \|^2_{\mathbf{Z}} \leq C e^{-\delta t} E(0)$$

which is exactly our uniform stabilization result. □

APPENDIX A
BASIC IDENTITIES

Let f, g be scalar functions, and $h(x) \in C^2(\overline{\Omega})$ vector field. The following identities will be needed:

$$\text{Green's Second Theorem } \int_\Omega \Delta f g d\Omega - \int_\Omega \Delta g f d\Omega = \int_\Gamma \frac{\partial f}{\partial \nu} g d\Gamma - \int_\Gamma \frac{\partial g}{\partial \nu} f d\Gamma \tag{A.0}$$

$$\text{Green's First Theorem } \int_\Omega \Delta f g d\Omega = \int_\Gamma \frac{\partial f}{\partial \nu} g d\Gamma - \int_\Omega \nabla f \cdot \nabla g d\Omega \tag{A.1}$$

$$\text{Divergence Theorem } \int_\Omega \text{div} h d\Omega = \int_\Gamma h \cdot \nu d\Gamma \tag{A.2}$$

$$\text{div}(fh) = h \cdot \nabla f + f \text{div} h \tag{A.3}$$

$$fh \cdot \nabla f = \frac{1}{2} h \cdot \nabla (f^2) \tag{A.4}$$

$$\int_\Omega h \cdot \nabla f d\Omega = \int_\Omega (\text{div} fh - f \text{div} h) d\Omega = \int_\Gamma fh \cdot \nu d\Gamma - \int_\Omega f \text{div} h d\Omega \tag{A.5}$$

$$\nabla f \cdot \nabla (h \cdot \nabla f) = H \nabla f \cdot \nabla f + \frac{1}{2} h \cdot \nabla (|\nabla f|^2) \tag{A.6}$$

$$\text{where } H(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix}$$

$$\nabla f \cdot \nabla (f \text{div} h) = f \nabla (\text{div} h) \cdot \nabla f + |\nabla f|^2 \text{div} h \tag{A.7}$$

$$\begin{aligned} \int_\Omega \Delta fh \cdot \nabla f d\Omega = & \int_\Gamma \frac{\partial f}{\partial \nu} h \cdot \nabla f d\Gamma - \frac{1}{2} \int_\Gamma |\nabla f|^2 h \cdot \nu d\Gamma \\ & - \int_\Omega H \nabla f \cdot \nabla f d\Omega + \frac{1}{2} \int_\Omega |\nabla f|^2 \text{div} h d\Omega \end{aligned} \tag{A.8}$$

Proof of (A.8):

Multiply Δf by $h \cdot \nabla f$, integrate $\int_\Omega d\Omega$, and successively apply (A.1), (A.6), and (A.5).

$$\int_\Omega \Delta f h \cdot \nabla f d\Omega = \int_\Gamma \frac{\partial f}{\partial \nu} h \cdot \nabla f d\Gamma - \int_\Omega \nabla f \cdot \nabla (h \cdot \nabla f) d\Omega$$

$$= \int_\Gamma \frac{\partial f}{\partial \nu} h \cdot \nabla f d\Gamma - \int_\Omega H \nabla f \cdot \nabla f d\Omega - \frac{1}{2} \int_\Omega h \cdot \nabla (|\nabla f|^2) d\Omega$$

$$= \int_\Gamma \frac{\partial f}{\partial \nu} h \cdot \nabla f d\Gamma - \int_\Omega H \nabla f \cdot \nabla f d\Omega - \frac{1}{2} \int_\Gamma |\nabla f|^2 h \cdot \nu d\Gamma + \frac{1}{2} \int_\Omega |\nabla f|^2 \mathrm{div} h d\Omega$$

Another useful identity arises by applying (A.1), (A.7) to get:

$$\int_\Omega \Delta f f \mathrm{div} h d\Omega = \int_\Gamma \frac{\partial f}{\partial \nu} f \mathrm{div} h d\Gamma - \int_\Omega \nabla f \cdot \nabla (f \mathrm{div} h) d\Omega$$

$$= \int_\Gamma \frac{\partial f}{\partial \nu} f \mathrm{div} h d\Gamma - \int_\Omega f \nabla (\mathrm{div} h) \cdot \nabla f d\Omega - \int_\Omega |\nabla f|^2 \mathrm{div} h d\Omega \qquad \text{(A.9)}$$

APPENDIX B
TO HANDLE DIFFERENCE OF ENERGY TERM

We multiply both sides of our equation $p_{tt} + \Delta^2 p = -A^{-1/2} G_1 G_1^* A p_t$ by $e^{-2\beta t} \Delta p \mathrm{div} h$ with $0<\beta\leq 1$ and integrate both sides over Q: $\int_Q dQ = \int_\Omega \int_0^\infty dt d\Omega$.

p_{tt} term: First integrate by parts in time, and then apply (A.9) to get:

$$\begin{aligned}
&\int_\Omega \int_0^\infty p_{tt} e^{-2\beta t} \Delta p \mathrm{div} h dt d\Omega = \lim_{T\to\infty} \Big[e^{-2\beta t} < p_t ,\ \Delta p \mathrm{div} h >_\Omega \Big]_0^T \\
&+ 2\beta \int_Q e^{-2\beta t} \Delta p p_t \mathrm{div} h dQ - \int_Q e^{-2\beta t} \Delta p_t p_t \mathrm{div} h dQ \\
&= \lim_{T\to\infty} \Big[e^{-2\beta t} < p_t ,\ \Delta p \mathrm{div} h >_\Omega \Big]_0^T + 2\beta \int_Q e^{-2\beta t} \Delta p p_t \mathrm{div} h dQ \\
&- \int_\Sigma e^{-2\beta t} \frac{\partial p_t}{\partial \nu} p_t \mathrm{div} h d\Sigma + \int_Q e^{-2\beta t} p_t \nabla(\mathrm{div} h) \cdot \nabla p_t dQ + \int_Q e^{-2\beta t} |\nabla p_t|^2 \mathrm{div} h dQ
\end{aligned} \tag{B.1}$$

$\Delta^2 p$ term : First apply (A.1) and then (A.7) to get:

$$\begin{aligned}
&\int_Q e^{-2\beta t} \Delta(\Delta p) \Delta p \mathrm{div} h dQ = \int_\Sigma e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} \Delta p \mathrm{div} h d\Sigma - \int_Q e^{-2\beta t} \nabla(\Delta p) \cdot \nabla(\Delta p \mathrm{div} h) dQ \\
&= \int_\Sigma e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} \Delta p \mathrm{div} h d\Sigma - \int_Q e^{-2\beta t} \Delta p \nabla(\mathrm{div} h) \cdot \nabla(\Delta p) dQ - \int_Q e^{-2\beta t} |\nabla(\Delta p)|^2 \mathrm{div} h dQ
\end{aligned} \tag{B.2}$$

Now isolating difference of energy term and multiplying by $\frac{1}{2}$ we get:

$$\begin{aligned}
&\frac{1}{2} \int_Q e^{-2\beta t} \Big[|\nabla p_t|^2 - |\nabla(\Delta p)|^2 \Big] \mathrm{div} h dQ = -\frac{1}{2} \lim_{T\to\infty} \Big[e^{-2\beta t} < p_t ,\ \Delta p \mathrm{div} h >_\Omega \Big]_0^T \\
&- \beta \int_Q e^{-2\beta t} \Delta p p_t \mathrm{div} h dQ + \frac{1}{2} \int_\Sigma e^{-2\beta t} \frac{\partial p_t}{\partial \nu} p_t \mathrm{div} h d\Sigma - \frac{1}{2} \int_Q e^{-2\beta t} p_t \nabla(\mathrm{div} h) \cdot \nabla p_t dQ \\
&- \frac{1}{2} \int_\Sigma e^{-2\beta t} \frac{\partial(\Delta p)}{\partial \nu} \Delta p \mathrm{div} h d\Sigma + \frac{1}{2} \int_Q e^{-2\beta t} \Delta p \nabla(\mathrm{div} h) \cdot \nabla p_t dQ \\
&- \frac{1}{2} \int_Q e^{-2\beta t} A^{-1/2} G_1 G_1^* A p_t \Delta p \mathrm{div} h dQ
\end{aligned} \tag{B.3}$$

APPENDIX C
TO OBTAIN GENERAL IDENTITY

We multiply our equation $p_{tt} + \Delta^2 p = -A^{-1/2} G_1 G_1^* A p_t$ by $e^{-2\beta t} h \cdot \nabla(\Delta p)$ with $0 < \beta \leq 1$, and integrate both sides over Q as in Appendix B.

p_{tt} term: First integrate by parts in time, then successively apply (A.5) and (A.3) to get:

$$\int_\Omega \int_0^\infty p_{tt} e^{-2\beta t} h \cdot \nabla(\Delta p) dt d\Omega = \lim_{T\to\infty} \left[e^{-2\beta t} < p_t ,\ h \cdot \nabla(\Delta p) >_\Omega \right]_0^T$$

$$+ 2\beta \int_Q e^{-2\beta t} p_t h \cdot \nabla(\Delta p) dQ - \int_Q e^{-2\beta t} p_t h \cdot \nabla(\Delta p_t) dQ$$

$$= \lim_{T\to\infty} \left[e^{-2\beta t} < p_t ,\ h \cdot \nabla(\Delta p) >_\Omega \right]_0^T + 2\beta \int_Q e^{-2\beta t} p_t h \cdot \nabla(\Delta p) dQ$$

$$- \int_\Sigma e^{-2\beta t} \Delta p_t p_t h \cdot \nu d\Sigma + \int_Q e^{-2\beta t} \Delta p_t \mathrm{div}(p_t h) dQ$$

$$= \lim_{T\to\infty} \left[e^{-2\beta t} < p_t ,\ h \cdot \nabla(\Delta p) >_\Omega \right]_0^T + 2\beta \int_Q e^{-2\beta t} p_t h \cdot \nabla(\Delta p) dQ - \int_\Sigma e^{-2\beta t} \Delta p_t p_t h \cdot \nu d\Sigma$$

$$+ \int_Q e^{-2\beta t} \Delta p_t h \cdot \nabla p_t dQ + \int_Q e^{-2\beta t} \Delta p_t p_t \mathrm{div} h dQ \qquad \text{(C.1)}$$

Next, we will apply (A.8) to $\int_Q e^{-2\beta t} \Delta p_t h \cdot \nabla p_t dQ$ and (A.9) to $\int_Q e^{-2\beta t} \Delta p_t p_t \mathrm{div} h dQ$ simultaneously to obtain:

$$\int_\Omega \int_0^\infty p_{tt} e^{-2\beta t} h \cdot \nabla(\Delta p) dt d\Omega$$

$$= \lim_{T\to\infty} \left[e^{-2\beta t} < p_t ,\ h \cdot \nabla(\Delta p) >_\Omega \right]_0^T + 2\beta \int_Q e^{-2\beta t} p_t h \cdot \nabla(\Delta p) dQ - \int_\Sigma e^{-2\beta t} \Delta p_t p_t h \cdot \nu d\Sigma$$

$$+ \int_\Sigma e^{-2\beta t} \frac{\partial p_t}{\partial \nu} h \cdot \nabla p_t d\Sigma - \frac{1}{2} \int_\Sigma e^{-2\beta t} |\nabla p_t|^2 h \cdot \nu d\Sigma - \int_Q e^{-2\beta t} H \nabla p_t \cdot \nabla p_t dQ + \frac{1}{2} \int_Q e^{-2\beta t} |\nabla p_t|^2 \mathrm{div} h dQ$$

$$+ \int_\Sigma e^{-2\beta t} \frac{\partial p_t}{\partial \nu} p_t \mathrm{div} h d\Sigma - \int_Q e^{-2\beta t} p_t \nabla(\mathrm{div} h) \cdot \nabla p_t dQ - \int_Q e^{-2\beta t} |\nabla p_t|^2 \mathrm{div} h dQ \qquad \text{(C.2)}$$

$\Delta^2 p$ term: Applying (A.8) we get:

$$\int_Q e^{-2\beta t}\Delta(\Delta p)h\cdot\nabla(\Delta p)dQ=\int_\Sigma e^{-2\beta t}\frac{\partial(\Delta p)}{\partial\nu}h\cdot\nabla(\Delta p)d\Sigma$$

$$-\frac{1}{2}\int_\Sigma e^{-2\beta t}|\nabla(\Delta p)|^2h\cdot\nu d\Sigma-\int_Q e^{-2\beta t}H\nabla(\Delta p)\cdot\nabla(\Delta p)dQ+\frac{1}{2}\int_Q e^{-2\beta t}|\nabla(\Delta p)|^2\mathrm{div}h dQ \quad (C.3)$$

Now we put all boundary terms on the left side, all interior terms on the right side and simplify to obtain:

$$\begin{aligned}
&-\int_\Sigma e^{-2\beta t}\Delta p_t p_t h\cdot\nu d\Sigma+\int_\Sigma e^{-2\beta t}\frac{\partial p_t}{\partial\nu}h\cdot\nabla p_t d\Sigma-\frac{1}{2}\int_\Sigma e^{-2\beta t}|\nabla p_t|^2h\cdot\nu d\Sigma\\
&+\int_\Sigma e^{-2\beta t}\frac{\partial p_t}{\partial\nu}p_t\mathrm{div}h d\Sigma+\int_\Sigma e^{-2\beta t}\frac{\partial(\Delta p)}{\partial\nu}h\cdot\nabla(\Delta p)d\Sigma-\frac{1}{2}\int_\Sigma e^{-2\beta t}|\nabla(\Delta p)|^2h\cdot\nu d\Sigma\\
&=-\lim_{T\to\infty}\left[e^{-2\beta t}<p_t,\ h\cdot\nabla(\Delta p)>_\Omega\right]_0^T-2\beta\int_Q e^{-2\beta t}p_t h\cdot\nabla(\Delta p)dQ\\
&+\int_Q e^{-2\beta t}H\nabla(\Delta p)\cdot\nabla(\Delta p)dQ+\int_Q e^{-2\beta t}H\nabla p_t\cdot\nabla p_t dQ\\
&+\int_Q e^{-2\beta t}p_t\nabla(\mathrm{div}h)\cdot\nabla p_t dQ+\frac{1}{2}\int_Q e^{-2\beta t}\left[|\nabla p_t|^2-|\nabla(\Delta p)|^2\right]\mathrm{div}h dQ\\
&-\int_Q e^{-2\beta t}A^{-1/2}G_1G_1^*Ap_t h\cdot\nabla(\Delta p)dQ
\end{aligned} \quad (C.4)$$

Before proceeding we make some simplifications which arise due to the boundary conditions.

Since $p\Big|_\Sigma=\frac{\partial p}{\partial\nu}\Big|_\Sigma=0$ it is immediate that

$$p_t\Big|_\Sigma=\frac{\partial p_t}{\partial\nu}\Big|_\Sigma=0 \quad (C.5)$$

Also $\nabla p\perp\Gamma$ and $\nabla p_t\perp\Gamma$ imply $|\nabla p|=|\frac{\partial p}{\partial\nu}|\equiv 0$ on Γ

$$\text{and } |\nabla p_t|=|\frac{\partial p_t}{\partial\nu}|\equiv 0 \text{ on } \Gamma \quad (C.6)$$

And by utilizing $h(x)=(x-x_0)$ the radial vector field, we have that

$$\begin{cases} H(x)=1 \quad \text{identity matrix}\\ \mathrm{div}h=n=\dim\Omega\\ \nabla(\mathrm{div}h)\equiv 0\end{cases} \quad (C.7)$$

Now inserting (B.3) for the difference of energy term in (C.4) and utilizing the simplifications (C.5), (C.6) and (C.7), we finally arrive at the desired identity:

$$
\begin{aligned}
&\int_\Sigma e^{-2\beta t}\frac{\partial(\Delta p)}{\partial \nu}h\cdot\nabla(\Delta p)d\Sigma-\frac{1}{2}\int_\Sigma e^{-2\beta t}|\nabla(\Delta p)|^2 h\cdot\nu d\Sigma+\frac{n}{2}\int_\Sigma e^{-2\beta t}\frac{\partial(\Delta p)}{\partial\nu}\Delta p d\Sigma\\
&=-\lim_{T\to\infty}\Big[e^{-2\beta t}<p_t,\ h\cdot\nabla(\Delta p)>_\Omega\Big]_0^T-\frac{n}{2}\lim_{T\to\infty}\Big[e^{-2\beta t}<p_t,\ \Delta p>_\Omega\Big]_0^T\\
&-2\beta\int_Q e^{-2\beta t}p_t h\cdot\nabla(\Delta p)dQ-n\beta\int_Q e^{-2\beta t}p_t\Delta p dQ\\
&+\int_Q e^{-2\beta t}|\nabla p_t|^2 dQ+\int_Q e^{-2\beta t}|\nabla(\Delta p)|^2 dQ\\
&-\int_Q e^{-2\beta t}A^{-1/2}G_1G_1^*Ap_t h\cdot\nabla(\Delta p)dQ-\frac{n}{2}\int_Q e^{-2\beta t}A^{-1/2}G_1G_1^*Ap_t\Delta p dQ
\end{aligned}
\tag{C.8}
$$

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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